We say that a sequence $\{s_n\}$ converges to L,

$$\lim_{n\to\infty} s_n = L$$

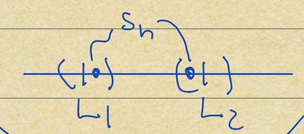
if $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $|s_n - L| < \varepsilon$ if $n > N$.

Thm: The limit of a convergent sequence is unique.

Proof: Suppose that, on the contrary, $\{s_n\}$ converges to two different limits L_1 and $L_2 \Rightarrow$

$$\lim_{n\to\infty} s_n = L_1 \quad \text{and} \quad \lim_{n\to\infty} s_n = L_2 \ , \quad L_1 \neq L_2$$

Motivation:



Since $L_1 \neq L_2$, then $|L_2 - L_1|$ is a finite $\# > 0$. Select $\varepsilon = \frac{|L_2 - L_1|}{4} \Rightarrow$ By definition of the limit, there exits an $N_1 \in \mathbb{R}$ s.t. $|s_n - L_1| < \varepsilon$, if $n > N_1$. Similarly, by the definition of the limit, there exists an $N_2 \in \mathbb{R}$ s.t. $|s_n - L_2| < \varepsilon$ if $n > N_2$. Now set $N = \max(N_1, N_2)$.

$$\text{If } n > N \Rightarrow \begin{cases} n > N_1 \Rightarrow |s_n - L_1| < \varepsilon \\ n > N_2 \Rightarrow |s_n - L_2| < \varepsilon \end{cases}$$

$$\Rightarrow |L_2 - L_1| = |L_2 - s_n + s_n - L_1| \leq |L_2 - s_n| + |s_n - L_1|$$

$$= |s_n - L_2| + |s_n - L_1| < \varepsilon + \varepsilon = 2\varepsilon$$

$$= 2\,\frac{|L_2 - L_1|}{4} = \frac{|L_2 - L_1|}{2} \quad \text{if } n > N$$

$\Rightarrow |L_2 - L_1| < \frac{|L_2 - L_1|}{2}$ – contradiction.

$\Rightarrow L_2 = L_1$ – the limit is unique.

Ex: $\lim\limits_{n \to \infty} \frac{1}{n} = 0$ – prove this by using the definition.

Choose any $\varepsilon > 0$. Want: $\left|\frac{1}{n} - 0\right| < \varepsilon$

$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon} \Rightarrow$ set $N = \frac{1}{\varepsilon}$.

Check: let $n > N \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow$

$\left|\frac{1}{n} - 0\right| < \varepsilon$, that is the definition of the limit checks out.

Ex: $\lim\limits_{n \to \infty} \frac{n}{2n+3} = \frac{1}{2}$ – prove this by using the def. of the limit.

(1) Choose any $\varepsilon > 0$; want: $\left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon$

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| = \left|\frac{2n - (2n+3)}{2(2n+3)}\right| = \left|\frac{2n - 2n - 3}{2(2n+3)}\right|$$

$$= \left| \frac{-3}{2(2n+3)} \right| = \frac{3}{2(2n+3)} < \varepsilon \Rightarrow \frac{2(2n+3)}{3} > \frac{1}{\varepsilon}$$

$$\Rightarrow 2n+3 > \frac{3}{2\varepsilon} \Rightarrow n > \frac{1}{2}\left(\frac{3}{2\varepsilon} - 3\right) = N$$

Check: Let $n > N = \frac{1}{2}\left(\frac{3}{2\varepsilon} - 3\right) \Rightarrow \left| \frac{n}{2n+3} - \frac{1}{2} \right| < \varepsilon$

– the definition of the limit is satisfied.

Ex: $\lim\limits_{n\to\infty} \frac{n^2-3}{n^4+2} = 0$ – prove this by using the definition of the limit.

(1) Choose any $\varepsilon > 0$. Want: $\left| \frac{n^2-3}{n^4+2} - 0 \right| < \varepsilon$

$$\left| \frac{n^2-3}{n^4+2} - 0 \right| \overset{n\geq 2}{=} \frac{n^2-3}{n^4+2} < \frac{n^2}{n^4+2} < \frac{n^2}{n^4} = \frac{1}{n^2} < \varepsilon \Leftrightarrow$$

$$n^2 > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\sqrt{\varepsilon}} = N$$

(2) If $n > N \Rightarrow \left| \frac{n^2-3}{n^4+2} - 0 \right| < \varepsilon$ – the definition of limit holds.

Ex: Show that the sequence $(-1)^n$ does not converge: Suppose, on the contrary, that

$$\lim_{n\to\infty} (-1)^n = L \Rightarrow$$

$\forall \varepsilon > 0, \exists N \in \mathbb{R}$ s.t. $n > N \Rightarrow |(-1)^n - L| < \varepsilon$

$\Rightarrow$ (1) there are odd $n > N$: $|(-1)^n - L| < \varepsilon$, hence

$$|-1-L| < \varepsilon \Rightarrow |L+1| < \varepsilon$$

(2) there are even $n > N$: $|(-1)^n - L| < \varepsilon$, hence

$$|1-L| < \varepsilon \Rightarrow |L-1| < \varepsilon$$

$\Rightarrow$ $|L-1| < \varepsilon$, $|L+1| < \varepsilon \Rightarrow 2 = |1-(-1)| = |1-L+L+1|$

$\overset{T.I.}{\leq} |1-L| + |L+1| = |L-1| + |L+1| < \varepsilon + \varepsilon = 2\varepsilon \Rightarrow$

$2 < 2\varepsilon$ for any $\varepsilon > 0$ - clearly false, e.g., if $\varepsilon = \frac{1}{2} \Rightarrow 2 < 1$ - contradiction, the limit $\lim\limits_{n\to\infty} (-1)^n$ does not exist.

Ex. Suppose that $\{s_n\}$ is a sequence of nonnegative #'s s.t. $\lim\limits_{n\to\infty} s_n = s$. Prove that $\lim\limits_{n\to\infty} \sqrt{s_n} = \sqrt{s}$.

(1) Show that $s \geq 0$: Suppose, on the contrary, that $s = a < 0 \Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{R}$ s.t. $n > N \Rightarrow$ $|s_n - a| < \varepsilon \Rightarrow |a| \leq |s_n - a| < \varepsilon$ - contradiction - just take $\varepsilon = \frac{|a|}{2}$. $\therefore s \geq 0$

(2) Suppose that $s = 0$: $\lim\limits_{n\to\infty} s_n = 0$. Show $\lim\limits_{n\to\infty} \sqrt{s_n} = 0$. Choose any $\varepsilon > 0$; want

$|\sqrt{s_n} - 0| < \varepsilon \iff |\sqrt{s_n}| < \varepsilon \iff \sqrt{s_n} < \varepsilon \iff$

$s_n < \varepsilon^2$.

By definition of limit $\lim_{n\to\infty} s_n = 0$, there exists an $N \in \mathbb{R}$ s.t. $n > N \Rightarrow |s_n - 0| < \varepsilon^2 \iff s_n < \varepsilon^2$

$\overset{\text{follow}}{\underset{\text{the chain of ineq.}}{\Longrightarrow}}$ $|\sqrt{s_n} - 0| < \varepsilon$ $\boxtimes$

(3) Suppose $s > 0$. Choose any $\varepsilon > 0$. Want:

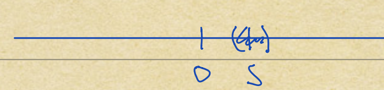
$|\sqrt{s_n} - \sqrt{s}| < \varepsilon$. We have:

$$\sqrt{s_n} - \sqrt{s} = \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} = \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}} \leq \frac{1}{\sqrt{s}}(s_n - s)$$

$$\Rightarrow |\sqrt{s_n} - \sqrt{s}| \leq \frac{1}{\sqrt{s}} |s_n - s| < \varepsilon \iff |s_n - s| < \varepsilon\sqrt{s}$$

By definition of limit $\lim_{n\to\infty} s_n = s$, there exists an $N \in \mathbb{R}$ s.t. $|s_n - s| < \varepsilon\sqrt{s}$ $\boxtimes$

<u>Ex:</u> Suppose that $\{s_n\}$ is a sequence of real #'s s.t. $s_n \neq 0$ for any $n \in \mathbb{N}$. If $\lim_{n\to\infty} s_n = s \neq 0$, show that $\inf\{|s_n|, n \in \mathbb{N}\} > 0$.

<u>Proof:</u> By definition of the limit, 

$\forall \varepsilon > 0$, there is an $N \in \mathbb{R}$ s.t. $n > N \Rightarrow |s_n - s| < \varepsilon$

Choose $\varepsilon = \frac{|s|}{2} \Rightarrow \exists N \in \mathbb{R}$ s.t. $n > N \Rightarrow |s_n - s| < \frac{|s|}{2}$

We have: $|s_n - s| \geq |s_n| - |s|$

$|s_n - s| \geq |s| - |s_n|$

$|a+b| \leq |a| + |b|$

set $c = a+b$

$\Rightarrow b = c - a \Rightarrow$

$|c| \leq |a| + |c-a|$

$\Rightarrow |c-a| \geq |c| - |a|$

Now combine $|s| - |s_n| \leq |s_n - s| < \frac{|s|}{2}$

if $n > N \Rightarrow \frac{|s|}{2} \leq |s_n|$ if $n > N$.

$\Rightarrow$ Consider the set: $\{|s_n|, n > N\}$;

$\frac{|s|}{2}$ is a lower bound for this set.

Now consider the set $\{|s_n|, 1 \leq n \leq N\}$ – this set is finite $\Rightarrow$ minimum exists $= \min\{|s_1|, |s_2|, \ldots, |s_N|\}$

Finally consider the set $\{|s_n|, n \geq 1\}$. The #

$A = \min\left\{\frac{|s|}{2}, \min\{|s_1|, \ldots, |s_N|\}\right\} = \min\{|s_1|, |s_2|, \ldots, |s_N|, \frac{|s|}{2}\}$ is a lower bound on $\{|s_n|, n \geq 1\}$.

A is strictly positive, $A > 0$, because it is a minimum of $N+1$ positive #'s. Therefore $\{s_n, n \geq 1\}$ has a positive lower bound $\Rightarrow$

its greatest lower bound must be positive. $\boxtimes$