<u>Thm:</u> Every convergent sequence (s_n) is bounded.

<u>Proof:</u> Suppose that $\lim_{n\to\infty} s_n = s$. Choose any $\varepsilon > 0$, then there exists a $N \in \mathbb{R}$ s.t. $|s_n - s| < \varepsilon$ if $n > N$.

Then $-\varepsilon < s_n - s < \varepsilon$ if $n > N \Rightarrow s - \varepsilon < s_n < s + \varepsilon$ if $n > N$. $\Rightarrow$ if $S_N = \{s_{N+1}, s_{N+2}, \dots\}$, then $s - \varepsilon$ is a lower bound on S_N and $s + \varepsilon$ is an upper bound on S_N.

Observe that $\{s_1, s_2, \dots\} = \{s_1, s_2, \dots, s_N\} \cup S_N$

Since $\{s_1, \dots, s_N\}$ is finite, it is bounded from below by $\min\{s_1, \dots, s_N\}$ and from above by the $\max\{s_1, \dots, s_N\} \Rightarrow \max\{s_1, \dots, s_N, s+\varepsilon\}$ is an upper bound on $\{s_1, s_2, \dots\}$ and $\min\{s_1, \dots, s_N, s-\varepsilon\}$ is a lower bound on $\{s_1, s_2, \dots\}$. $\boxtimes$

<u>Thm:</u> Suppose that $\lim_{n\to\infty} s_n = s$ and $k \in \mathbb{R}$. Then (ks_n) converges as $n \to \infty$ and $\lim_{n\to\infty} ks_n = k \lim_{n\to\infty} s_n$

<u>Idea:</u> $|ks_n - ks| = |k(s_n - s)| = |k||s_n - s_n| < \varepsilon$

$\Leftrightarrow |s_n - s| < \frac{\varepsilon}{|k|}$ (assuming that $k \neq 0$)

<u>Proof:</u> Let $k \neq 0$. Choose any $\varepsilon > 0$, then (by def of limit $\lim_{n\to\infty} s_n = s$), there exists a $N \in \mathbb{R}$ s.t. $|s_n - s| < \frac{\varepsilon}{|k|}$ if $n > N$.

It follows that, if $n > N$: $|k||s_n - s| < \varepsilon \Rightarrow$

$\Rightarrow |k(s_n - s)| < \varepsilon \Rightarrow |ks_n - ks| < \varepsilon \Rightarrow$ definition of the limit holds for ks_n and

$$\lim_{n\to\infty} ks_n = ks$$

Let $k = 0$: Choose any $\varepsilon > 0$ and let $N = 0 \Rightarrow$ if $n > N$, $|ks_n - ks| = |0 \cdot s_n - 0 \cdot s| = 0 < \varepsilon \Rightarrow$

$$\lim_{n\to\infty} ks_n = ks \qquad \boxtimes$$

<u>Thm:</u> Suppose that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then

$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t$$

<u>Idea:</u> $|(s_n + t_n) - (s+t)| = |s_n + t_n - s - t| = |(s_n - s) + (t_n - t)|$

$\overset{T.I.}{\leq} |s_n - s| + |t_n - t| < \varepsilon \leftarrow$ can require $|s_n - s| < \frac{\varepsilon}{2}$, $|t_n - t| < \frac{\varepsilon}{2}$.

<u>Proof:</u> Choose any $\varepsilon > 0$. (1) Because $\lim_{n\to\infty} s_n = s$,

$\exists N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$

(2) Because $\lim_{n\to\infty} t_n = t$,

$\exists N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$.

Now let $n > N = \max\{N_1, N_2\} \Rightarrow n > N_1$, hence

$|s_n - s| < \frac{\varepsilon}{2}$ and $n > N_2$, hence $|t_n - t| < \frac{\varepsilon}{2}$. Therefore, if $n > N$, $|(s_n+t_n)-(s+t)| = |s_n+t_n-s-t|$

$= |(s_n-s)+(t_n-t)| \overset{T.I.}{\leq} |s_n-s|+|t_n-t| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ ⊠

__Thm:__ If $\lim\limits_{n\to\infty} s_n = s$ and $\lim\limits_{n\to\infty} t_n = t$, then

$\lim\limits_{n\to\infty} s_n t_n = st$

__Idea:__ $|s_nt_n - st| = |s_nt_n - s_nt + s_nt - st|$

$= |s_n(t_n-t) + (s_n-s)t| \overset{T.I.}{\leq} |s_n(t_n-t)| + |t(s-s_n)|$

$= |s_n||t-t_n| + |t||s-s_n| \overset{|s_n|<M,\,\forall n}{\leq} M|t_n-t| + |t||s-s_n|$

$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ if $|t_n-t| < \frac{\varepsilon}{2M}$ and $|s_n-s| < \frac{\varepsilon}{2|t|}$.

__Proof:__ By the first thm above, because $\{s_n\}$ converges, it is bounded $\Rightarrow$ there exists an $M>0$ s.t. $|s_n| < M$ for all $n \geq 1$. Set $\bar{M} = \max\{M, |t|\}$

Choose any $\varepsilon > 0$. (1) Because $\lim\limits_{n\to\infty} s_n = s$, there exists an $N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2\bar{M}}$

(2) Because $\lim\limits_{n\to\infty} t_n = t$, there exists an $N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2\bar{M}} \Rightarrow$ If $n > \max\{N_1, N_2\}$ then $n > N_1$ and $|s_n - s| < \frac{\varepsilon}{2\bar{M}}$; also $n > N_2$ and $|t_n - t| < \frac{\varepsilon}{2\bar{M}}$

$$\Rightarrow |s_n t_n - st| = |s_n t_n - s_n t + s_n t - st|$$

$$= |s_n(t_n - t) + (s_n - s)t| \overset{\text{T.I.}}{\leq} |s_n(t_n - t)| + |t(s - s_n)|$$

$$= |s_n||t - t_n| + |t||s - s_n| \overset{|s_n| < M,\, \forall n}{\leq} M|t_n - t| + |t||s - s_n|$$

$$\leq \bar{M}|t_n - t| + \bar{M}|s - s_n| < \cancel{\bar{M}}\frac{\varepsilon}{2\cancel{\bar{M}}} + \cancel{\bar{M}}\frac{\varepsilon}{2\cancel{\bar{M}}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \boxtimes$$

Recall that, if $\{s_n\}$ is such that $s_n \neq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} s_n = s \neq 0 \Rightarrow \inf|s_n| > 0$.

Theorem: Suppose that $\{s_n\}$ is such that $s_n \neq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} s_n = s \neq 0$, then

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$$

<u>Idea:</u> For an $\varepsilon > 0$, want $\left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$;

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s_n - s|}{|s_n||s|} \leq \frac{|s_n - s|}{m|s|}, \text{ because,}$$

by the result above $|s_n| \geq \inf|s_n| = m > 0$ for all $n \in \mathbb{N}$.

$$\Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{|s_n - s|}{m|s|} < \varepsilon \iff |s_n - s| < \underbrace{m|s|}_{\text{constant}}\varepsilon$$

<u>Proof:</u> Choose any $\varepsilon > 0$; since $\lim_{n\to\infty} s_n = s$, by

definition of the limit, there exists an $N \in \mathbb{R}$ s.t. $|s_n - s| < m|s|\varepsilon$ if $n > N$. Then, if $n > N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s_n - s|}{|s_n||s|} \leq \frac{|s_n - s|}{m|s|} < \frac{\cancel{m|s|}\varepsilon}{\cancel{m|s|}} = \varepsilon \quad \boxtimes$$

Theorem: Suppose that $\{s_n\}$ is such that $s_n \neq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} s_n = s \neq 0$. Assume also that $\lim_{n\to\infty} t_n = t \Rightarrow$

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{t}{s} = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}$$

<u>Proof:</u> By the preceding thm, $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s} \Rightarrow$

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \lim_{n\to\infty} t_n \cdot \frac{1}{s_n} \overset{\text{Thm}}{=} \lim_{n\to\infty} t_n \lim_{n\to\infty} \frac{1}{s_n} = t\,\frac{1}{s}$$

$$= \lim_{n\to\infty} t_n \Big/ \lim_{n\to\infty} s_n \quad \boxtimes$$

<u>Thm:</u> Basic limits:

(1) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$

<u>Proof:</u> Choose any $\varepsilon > 0$ and set $N = \frac{1}{\varepsilon^{1/p}} \Rightarrow$

if $n > N$, we have $\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \frac{1}{N^p} = \frac{1}{(1/\varepsilon^{1/p})^p}$

$= \frac{1}{1/\varepsilon} = \varepsilon$, because $n > N \Rightarrow n^p > N^p \Rightarrow \frac{1}{n^p} < \frac{1}{N^p}$ $\boxtimes$

(2) $\lim\limits_{n\to\infty} a^n = 0$ if $|a| < 1$

<u>Proof:</u> (a) $a = 0 \Rightarrow$ trivial, because $|a|^n = 0$ for all n and the limit of a constant sequence is that constant.

(b) Suppose that $0 < |a| < 1 \Rightarrow |a| = \frac{1}{1+b}$, for some $b > 0$. Then $|a|^n = \left(\frac{1}{1+b}\right)^n$

$$= \frac{1}{(1+b)^n} \leq \frac{1}{1+nb} < \frac{1}{nb}.$$

Now, if $\varepsilon > 0$, set $N = \frac{1}{\varepsilon b} \Rightarrow$

$$\left||a|^n - 0\right| = |a|^n < \frac{1}{nb} \leq \frac{1}{\frac{1}{\varepsilon \cancel{b}}\cancel{b}} = \varepsilon$$

if $n > N$ ⊠

Prove that if $\underline{\underline{b > 0}}$

$$(1+b)^n \geq 1 + nb$$

Use induction:

(1) base step:

$(1+b)^1 = 1 + b$ – true

(2) Assume that $(1+b)^n > 1 + nb \Rightarrow$

$$(1+b)^{n+1} = (1+b)^n(1+b) > (1+nb)(1+b) =$$

$$= 1 + nb + b + \underline{\underline{nb^2}} > 1 + (n+1)b \ \square$$

(3) $\lim\limits_{n\to\infty} n^{1/n} = 1$

To prove this, need the following:

For any $a \geq 0$ and $n \in \mathbb{N}$

$$(1+a)^n \geq \frac{1}{2}n(n-1)a^2 + na + 1$$

To demonstrate this, use induction: (a) base step: $(1+a)^1 = 1 + a = \frac{1}{2}1\cdot(1-1)a^2 + 1\cdot a + 1$

(b) Inductive step: Assume that

$(1+a)^n \geq \frac{1}{2}n(n-1)a^2 + na + 1$ holds $\Rightarrow$

$$(1+a)^{n+1} = (1+a)(1+a)^n \geq (1+a)\left[\frac{1}{2}n(n-1)a^2 + na + 1\right]$$

$$= \frac{1}{2}n(n-1)a^2 + na + 1 + \frac{1}{2}n(n-1)a^3 + na^2 + a$$

$$= \frac{1}{2}n(n-1)a^2 + na^2 + (na+a) + 1 + \frac{1}{2}n(n-1)a^3$$

$$= \frac{1}{2}n(n+1)a^2 + (n+1)a + 1 + \frac{1}{2}n(n-1)a^3$$

$$> \frac{1}{2}n(n+1)a^2 + (n+1)a + 1 \quad \boxtimes$$

Proof of (3): Consider $s_n = n^{1/n} - 1 \geq 0 \Rightarrow$

$n^{1/n} = s_n + 1 \Rightarrow n = (s_n+1)^n \geq \frac{1}{2}n(n-1)s_n^2 + \underbrace{ns_n + 1}_{>0}$

$> \frac{1}{2}n(n-1)s_n^2 \Rightarrow n > \frac{1}{2}n(n-1)s_n^2 \Rightarrow$

$1 > \frac{1}{2}(n-1)s_n^2 \Rightarrow \frac{2}{n-1} > s_n^2 \Rightarrow s_n < \sqrt{\frac{2}{n-1}}$

We want to show that $\lim_{n\to\infty} s_n = 0$. To this end

Now choose any $\varepsilon > 0$. Want $|s_n - 0| < \varepsilon$ if $n > N$ for some $N \in \mathbb{R}$. We have

$$|s_n - 0| = |s_n| = s_n < \sqrt{\frac{2}{n-1}} < \varepsilon \Leftrightarrow \frac{2}{n-1} < \varepsilon^2$$

$\Leftrightarrow n - 1 > \frac{2}{\varepsilon^2} \Leftrightarrow n > \frac{2}{\varepsilon^2} + 1 = N \Rightarrow$ if $n > N$

following the chain of inequalities backward, we conclude that $|s_n - 0| < \varepsilon \Rightarrow \lim_{n\to\infty} s_n = 0$

by definition of the limit $\Rightarrow$

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} (\underbrace{n^{1/n} - 1}_{s_n} + 1) \overset{\text{Prop of}}{\underset{\text{limits}}{=}} \lim_{n\to\infty} s_n$$

$$+ \lim_{n\to\infty} 1 = 0 + 1 = 1 \quad \boxtimes$$

(4) $\lim_{n\to\infty} a^{1/n} = 1$ for any $a > 0$.

<u>Proof:</u> Consider the case $a \geq 1$ first. Then,

if $n > a \Rightarrow \quad 1 < a^{1/n} < n^{1/n} \Rightarrow \lim_{n\to\infty} a^{1/n} \geq 1$

and, $\lim_{n\to\infty} n^{1/n} = 1 \Rightarrow \lim_{n\to\infty} a^{1/n} \leq 1 \Rightarrow \lim_{n\to\infty} a^{1/n} = 1$

Now suppose $a < 1 \Rightarrow \frac{1}{a} > 1 \Rightarrow \lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n} = 1$

by above $\Rightarrow \quad 1 = \lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n} = \lim_{n\to\infty} \frac{(1)^{1/n}}{a^{1/n}}$

$$= \lim_{n\to\infty} \frac{1}{a^{1/n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} a^{1/n}} = \frac{1}{\lim_{n\to\infty} a^{1/n}} \Rightarrow \lim_{n\to\infty} a^{1/n} = 1 \quad \boxtimes$$

<u>Ex. 1:</u> $\lim_{n\to\infty} \left(\frac{5}{\sqrt{n}} + 4n^{-2}\right) = \lim_{n\to\infty} \frac{5}{\sqrt{n}} + \lim_{n\to\infty} 4n^{-2}$

$$= 5 \lim_{n\to\infty} \frac{1}{\sqrt{n}} + 4 \lim_{n\to\infty} \frac{1}{n^2} = 5 \cdot 0 + 4 \cdot 0 = 0$$

$\underline{Ex\ 1.5}$: $\lim\limits_{n\to\infty}\left(2n^{1/n}+4\cdot 5^{1/n}\right) = 2\lim\limits_{n\to\infty} n^{1/n} + 4\lim\limits_{n\to\infty} 5^{1/n}$

$= 2\cdot 1 + 4\cdot 1 = 6$

$\underline{Ex\ 2}$: $\lim\limits_{n\to\infty}\dfrac{n^2+5n+1}{2n^2-n+6} = \lim\limits_{n\to\infty}\dfrac{\frac{1}{n^2}(n^2+5n+1)}{\frac{1}{n^2}(2n^2-n+6)}$

$= \lim\limits_{n\to\infty}\dfrac{1+5/n+1/n^2}{2-1/n+6/n^2} = \dfrac{\lim\limits_{n\to\infty}(1+5/n+1/n^2)}{\lim\limits_{n\to\infty}(2-1/n+6/n^2)}$

$= \dfrac{\lim\limits_{n\to\infty}1 + 5\overbrace{\cancel{\lim\limits_{n\to\infty}\frac{1}{n}}}^{0} + \overbrace{\cancel{\lim\limits_{n\to\infty}\frac{1}{n^2}}}^{0}}{\lim\limits_{n\to\infty}2 - \overbrace{\cancel{\lim\limits_{n\to\infty}\frac{1}{n}}}^{0} + 6\overbrace{\cancel{\lim\limits_{n\to\infty}\frac{1}{n^2}}}^{0}} = \dfrac{1}{2}$

$\underline{Ex\ 3}$: $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-n\right) = \lim\limits_{n\to\infty}\dfrac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}$

$= \lim\limits_{n\to\infty}\dfrac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n} = \lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}+n} = \lim\limits_{n\to\infty}\dfrac{n/n}{\frac{1}{n}\left(\sqrt{n^2+n}+n\right)}$

$= \lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\frac{1}{n}}+1} = \lim\limits_{n\to\infty}1 \Big/ \lim\limits_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}+1\right)$

$= 1\Big/\left(\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}} + \lim\limits_{n\to\infty}1\right) = 1\Big/\left(\sqrt{\underbrace{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)}_{\lim\limits_{n\to\infty}1+\lim\limits_{n\to\infty}\frac{1}{n}=1+0}} + \lim\limits_{n\to\infty}1\right)$

$= \dfrac{1}{\sqrt{1}+1} = \dfrac{1}{2}$

Def: The sequence $\{s_n\}$ converges to ∞ if for any $M>0$, there exists an $N \in \mathbb{R}$ s.t. $s_n > M$ if $n > N$. In this case, we write:

$$\lim_{n\to\infty} s_n = \infty$$

Def: The sequence $\{s_n\}$ converges to $-\infty$ if for any $M<0$, there exists an $N \in \mathbb{R}$ s.t. $s_n < M$ if $n > N$. In this case, we write:

$$\lim_{n\to\infty} s_n = -\infty$$

Ex 1: $\lim\limits_{n\to\infty} 2n^2+3 = \infty$

Idea: Choose $M > 3$; want $2n^2+3 > M \iff$

$2n^2 > M-3 \iff n^2 > \frac{M-3}{2} \iff n > \sqrt{\frac{M-3}{2}} = N$

Proof: Suppose $n > N \iff n > \sqrt{\frac{M-3}{2}} \iff n^2 > \frac{M-3}{2}$

$\iff 2n^2 > M-3 \iff 2n^2+3 > M$ $\boxtimes$

Ex 2: $\lim\limits_{n\to\infty} \frac{n^3+1}{n^2-3} = \infty$

Choose $M>0$: Want $\frac{n^3+1}{n^2-3} > M$

Idea: $\frac{n^3+1}{n^2-3} > \frac{n^3}{n^2-3} > \frac{n^3}{n^2} = n > M = N$

Proof: Suppose $n > N \Rightarrow n > M$. Then

$$\frac{n^3+1}{n^2-3} > \frac{n^3}{n^2-3} > \frac{n^3}{n^2} = n > M \quad \boxtimes$$

<u>Ex 3</u> $\lim_{n\to\infty} \frac{2n^2-n^3}{n+6} = -\infty$

Choose any $M<0$; Want $\frac{2n^2-n^3}{n+6} < M$

Idea: $\frac{2n^2-n^3}{n+6} < \frac{\frac{n^3}{2}-n^3}{n+6}$ if $n>4$ because

$$2n^2 \overset{?}{<} \frac{n^3}{2} \Leftrightarrow 4n^2 < n^3 \Leftrightarrow 4<n$$

$$\Rightarrow \frac{2n^2-n^3}{n+6} < -\frac{n^3/2}{n+6} < -\frac{n^3/2}{n+n} \text{ as long as } n>6$$

$$\Rightarrow \text{If } n>6, \quad \frac{2n^2-n^3}{n+6} < \frac{-n^3/2}{2n} = -\frac{n^2}{4} < M$$

$$\Leftrightarrow n^2 > -4M \Leftrightarrow n > \sqrt{-4M}$$

works because $M<0$

$\Rightarrow N = \max(6, \sqrt{-4M}) \Rightarrow$ if $n>N$,

we have $\frac{2n^2-n^3}{n+6} < M \quad \boxtimes$

<u>Thm 1:</u> $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = t$, where $t=\infty$ or a finite positive #. Then $\lim_{n\to\infty} t_n s_n = \infty$.

<u>Proof:</u>

Idea: Given $M > 0$, want to show that there exists an $N \in \mathbb{R}$ s.t. $n > N \Rightarrow s_n t_n > M$

Proof:

(1) $\lim_{n\to\infty} t_n = \infty \Rightarrow \exists N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow t_n > 1$

Also, since $\lim_{n\to\infty} s_n = \infty$, there exists an $N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow s_n > M$. Then set $N = \max(N_1, N_2)$

$\Rightarrow$ if $n > N \Rightarrow t_n > 1$ and $s_n > M \Rightarrow s_n t_n > M$.

(2) $\lim_{n\to\infty} t_n = t$ - finite positive # $\Rightarrow$ there exists an $N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow |t_n - t| < \frac{t}{2} \Leftrightarrow$

$n > N_1$ implies $-\frac{t}{2} < t_n - t < \frac{t}{2}$ or $\frac{t}{2} < t_n < \frac{3t}{2}$

Therefore, if $n > N_1$, we have that $t_n > \frac{t}{2}$.

Because $\lim_{n\to\infty} s_n = \infty$, there exists an $N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow s_n > M/_{t/2} = \frac{2M}{t}$.

Then, if $N = \max\{N_1, N_2\}$ whenever $n > N$ we have that $t_n > t/2$ and $s_n > \frac{2M}{t}$

$\Rightarrow n > N$, then $s_n t_n > \frac{\cancel{t}}{\cancel{2}} \cdot \frac{\cancel{2}M}{\cancel{t}} = M$ $\boxtimes$

Thm 2: (1) Suppose that $\lim_{n\to\infty} s_n = \infty \Rightarrow$

$\lim_{n\to\infty} \frac{1}{s_n} = 0$.

(2) Suppose that $\{s_n\}$ is a sequence of positive #'s such that $\lim_{n\to\infty} s_n = 0 \Rightarrow$

$\lim_{n\to\infty} \frac{1}{s_n} = \infty$

Proof: (1) Choose $\varepsilon > 0$. Then, because $\lim_{n\to\infty} s_n = \infty$, there exists a $N \in \mathbb{R}$ s.t. $s_n > \frac{1}{\varepsilon}$ if $n > N$. Then $\left|\frac{1}{s_n} - 0\right| = \left|\frac{1}{s_n}\right|$

$= \frac{1}{s_n} < \frac{1}{1/\varepsilon} = \varepsilon$ ☒

(2) $\{s_n\}$ is a sequence of positive #'s s.t. $\lim_{n\to\infty} s_n = 0$. Choose any $M > 0$, then there exists an $N \in \mathbb{R}$ s.t. $|s_n - 0| < \frac{1}{M}$ if $n > N$

$\overset{s_n > 0}{\iff} s_n < \frac{1}{M}$ if $n > N \iff \frac{1}{s_n} > M$ if $n > N$ ☒