

Def: A sequence $\{s_n\}$ is increasing if $s_{n+1} \geq s_n$ for all $n \geq 1$;

Def: A sequence $\{s_n\}$ is decreasing if $s_{n+1} \leq s_n$ for all $n \geq 1$;

If a sequence is either decreasing or increasing then it is monotone;

If the inequalities in the definition above are replaced by strict inequalities, e.g., $s_{n+1} \geq s_n \Rightarrow s_{n+1} > s_n$, then a given sequence is strictly increasing or decreasing.

Ex 1: Suppose that $s_n = \frac{1}{n} \Rightarrow \forall n \geq 1, s_{n+1} - s_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n - (n+1)}{n(n+1)} = -\frac{1}{n(n+1)} < 0 \Rightarrow s_{n+1} - s_n < 0 \Rightarrow s_{n+1} < s_n \Rightarrow \{s_n\}$ is strictly decreasing.

Ex 2: Suppose that $s_n = n^2 \Rightarrow \forall n \geq 1, s_{n+1} - s_n = (n+1)^2 - n^2 = \cancel{n^2} + 2n+1 - \cancel{n^2} = 2n+1 > 0 \Rightarrow$

$s_{n+1} - s_n > 0 \Rightarrow s_{n+1} > s_n \Rightarrow$ the sequence is strictly increasing.

Ex. 3: Suppose that $s_n = (-1)^n \Rightarrow -1, 1, -1, 1,$

... - sequence is not monotone.

Ex. 4. (a) Suppose that $s_n = 1$ for all $n \geq 1 \Rightarrow$

$$s_{n+1} - s_n = 1 - 1 = 0, \forall n \geq 1 \Rightarrow s_{n+1} = s_n, \forall n \geq 1$$

$$\Rightarrow \begin{cases} s_{n+1} \leq s_n, \forall n \geq 1 & - \{s_n\} \text{ is decreasing} \\ s_{n+1} \geq s_n, \forall n \geq 1 & - \{s_n\} \text{ is increasing} \end{cases}$$

(b) Suppose that $\{s_n\}$ is both increasing and decreasing $\Rightarrow s_{n+1} \geq s_n, \forall n \geq 1$ and

$$s_{n+1} \leq s_n, \forall n \geq 1 \Rightarrow s_{n+1} = s_n, \forall n \geq 1 \Rightarrow$$

$\{s_n\}$ is a constant sequence.

Ex. 5 Is the sequence $s_n = \frac{2n+1}{n-2}, n \geq 3$

monotone? Consider $s_{n+1} - s_n = \frac{2(n+1)+1}{(n+1)-2} - \frac{2n+1}{n-2}$

$$= \frac{(2n+3)(n-2) - (n-1)(2n+1)}{(n-1)(n-2)} = \frac{\cancel{2n^2} - 4n + 3n - 6 - (\cancel{2n^2} + n - 2n - 1)}{(n-1)(n-2)}$$

$$= -\frac{5}{(n-1)(n-2)} < 0 \text{ if } n \geq 3 \Rightarrow s_{n+1} < s_n \text{ if } n \geq 3$$

$\Rightarrow \{s_n\}$ is decreasing.

Ex. 5. Is the sequence $s_n = e^{-n^2}$ monotone?

$$\begin{aligned}
 (1) \quad s_{n+1} - s_n &= e^{-(n+1)^2} - e^{-n^2} = \underbrace{e^{-n^2}}_{>0} (e^{-(n+1)^2 + n^2} - 1) \\
 &= e^{-n^2} (e^{-2n-1} - 1) = e^{-n^2} \underbrace{(e^{-2n-1} - 1)}_{<1} < 0
 \end{aligned}$$

$\Rightarrow s_{n+1} < s_n \quad \forall n \geq 1 \Rightarrow \{s_n\}$ is strictly $\downarrow$.

$$(2) \quad s_n = e^{-n^2} \Rightarrow \text{If } f(x) = e^{-x^2} \Rightarrow s_n = f(n);$$

Check if f is $\uparrow$ or $\downarrow$:

$$f'(x) = e^{-x^2} (-2x) = -2xe^{-x^2} < 0 \text{ if } x > 0$$

$\Rightarrow f$ is strictly $\downarrow$ when $x > 0 \Rightarrow \{s_n\}$ is strictly $\downarrow$.

Properties of monotone sequences:

Thm. 1: Any increasing sequence $\{s_n\}$ that is bounded from above converges.

Proof: Since $\{s_n\}$ is bounded from above $\{s_n\}$ has a supremum, $\sup s_n = L$. It follows that $s_n \leq L$ for all $n \geq 1 \Rightarrow s_n \leq L + \varepsilon$ for any $\varepsilon > 0$.

Fix an $\varepsilon > 0$: Because L is the supremum of $\{s_n\}$, $L - \varepsilon$ is not an upper bound on $\{s_n\} \Rightarrow$ there exists

an $s_N > L - \epsilon$. We know that $\{s_n\}$ is increasing
 $\Rightarrow s_n \geq s_N$ for any $n > N \Rightarrow s_n > L - \epsilon$ for all

$n > N \Rightarrow L - \epsilon < s_n < L + \epsilon$ if $n > N \Rightarrow$

$-\epsilon < s_n - L < \epsilon$ if $n > N \Rightarrow$

$|s_n - L| < \epsilon$ if $n > N \Rightarrow$

$\lim_{n \rightarrow \infty} s_n = L$ by the definition of the limit.

In the same way, we can prove

Thm 2: Any decreasing sequence $\{s_n\}$ that is bounded from below converges.

$\Rightarrow$ Any monotone, bounded sequence converges.

Thm 3. Suppose that $\{s_n\}$ is increasing and not bounded from above $\Rightarrow \lim_{n \rightarrow \infty} s_n = \infty$

Proof: Choose $M > 0 \Rightarrow$ This M is not an upper bound of $\{s_n\}$ because $\{s_n\}$ is not bounded from above $\Rightarrow$ there exists an $N \in \mathbb{N}$ s.t. $s_N > M$. However, $\{s_n\}$ is $\uparrow \Rightarrow s_n \geq s_N$ if $n > N \Rightarrow s_n > M$ if $n > N \Rightarrow \lim_{n \rightarrow \infty} s_n = \infty$ $\square$

Thm. 4. Suppose that $\{s_n\}$ is decreasing and not bounded from below $\Rightarrow \lim_{n \rightarrow \infty} s_n = -\infty$

$\Rightarrow$ Any monotone sequence $\{s_n\}$ converges.

Ex.: Suppose $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 2}$, $\forall n \geq 1$

$\Rightarrow s_1 = 1$; $s_2 = \sqrt{1+2} = \sqrt{3}$, $s_3 = \sqrt{\sqrt{3}+2}$, ...; s_n in terms of n is not available. Does this sequence converge and, if yes, what is the limit?

Conjecture: $\{s_n\}$ is bounded: $s_n < 2$, $\forall n$

To show this, use induction.

(a) Base step: $n=1 \Rightarrow s_1 = 1 < 2$.

(b) Inductive step: Suppose $s_n < 2 \Rightarrow s_n + 2 < 2 + 2 = 4$

$$\Rightarrow s_{n+1} = \sqrt{s_n + 2} < \sqrt{2 + 2} = 2. \quad \square$$

$\therefore \{s_n\}$ is bounded from above by 2.

Conjecture: $\{s_n\}$ is monotone $\uparrow$. To show this,

$$\begin{aligned} \text{fix any } n \geq 1 \Rightarrow s_{n+1} &= \sqrt{s_n + 2} > \sqrt{s_n + s_n} = \sqrt{2s_n} > \sqrt{s_n \cdot s_n} \\ &= \sqrt{s_n^2} = s_n \Rightarrow s_{n+1} > s_n \end{aligned}$$

$\therefore \{s_n\}$ is increasing.

$\Rightarrow \{s_n\}$, being monotone increasing and bounded from above, converges $\Rightarrow \lim_{n \rightarrow \infty} s_n$ exists and equal to some $s \in \mathbb{R}$. To find s ,

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_{n+1} = \lim_{n \rightarrow \infty} \sqrt{s_n + 2} = \sqrt{\lim_{n \rightarrow \infty} (s_n + 2)}$$

$$= \sqrt{s+2} \Rightarrow s = \sqrt{s+2} \Rightarrow s^2 = s+2 \Rightarrow$$

$$s^2 - s - 2 = 0 \Rightarrow \cancel{s = -1}, s = 2 \Rightarrow \boxed{\lim_{n \rightarrow \infty} s_n = 2}.$$

Given any sequence $\{s_n\}$ what are possible choices of behavior when $n \rightarrow \infty$.

(1) $\{s_n\}$ is monotone $\Rightarrow \lim_{n \rightarrow \infty} s_n = s \rightarrow s$ is either a number, ∞ , or $-\infty$.

(2) There exists an index $N \in \mathbb{N}$ s.t. the tail of the sequence $\{s_n\}_{n \geq N}$ is monotone. Then:

$\lim_{n \rightarrow \infty} s_n = s \rightarrow s$ is either a number, ∞ , or $-\infty$

(3) $\{s_n\}$ oscillates as $n \rightarrow \infty \Rightarrow \lim_{n \rightarrow \infty} s_n$ may or may not exist.

Ex. 1 $s_n = (-1)^n \Rightarrow \{s_n\}$ oscillates between -1 and 1 - does not converge.

Ex. 2 $s_n = \frac{(-1)^n}{n} \Rightarrow s_1 = -1, s_2 = \frac{1}{2}, s_3 = -\frac{1}{3} \dots$

- sequence oscillates but the window of oscillations, decreases to 0 as $n \rightarrow \infty \Rightarrow s_n \rightarrow 0$ as $n \rightarrow \infty$.

How to prove that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$?

Try to use properties of limits: $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$
 $= \lim_{n \rightarrow \infty} (-1)^n \lim_{n \rightarrow \infty} \frac{1}{n} = (\text{d.n.e.}) \cdot 0 = 0$ - makes no sense!!!

- cannot use properties!!!

One alternative is to use the definition of limit:

Idea: $|s_n - 0| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} = N$

$\Rightarrow$ if $n > N$, then $|s_n - 0| < \varepsilon$ $\square$

another alternative: use the Theorem

Thm: $\lim_{n \rightarrow \infty} s_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} |s_n| = 0.$

Then, $\lim_{n \rightarrow \infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0.$

How can we measure what happens to oscillations of $s_n = \frac{(-1)^n}{n}$ as n gets bigger?

Consider $\{s_n\}$ as a subset of $\mathbb{R}$. Thinking about oscillations, take $\sup\{s_n\}$ and $\inf\{s_n\} \Rightarrow$ all elements of $\{s_n\}$ lie between $\inf\{s_n\}$ and $\sup\{s_n\} \Rightarrow (\inf\{s_n\}, \sup\{s_n\})$ provides the "window" within which the entire sequence oscillates.

$$\inf_{n \geq 1} \frac{(-1)^n}{n} = -1 \quad \text{and} \quad \sup_{n \geq 1} \frac{(-1)^n}{n} = \frac{1}{2}$$

$$\text{since } s_1 = -1, s_2 = \frac{1}{2}, s_3 = -\frac{1}{3}, s_4 = \frac{1}{4}, \dots$$

$\Rightarrow$

$$\inf_{n \geq 2} \frac{(-1)^n}{n} = -\frac{1}{3} \quad \text{and} \quad \sup_{n \geq 2} \frac{(-1)^n}{n} = \frac{1}{2}$$

$\Rightarrow$

$$\inf_{n \geq 3} \frac{(-1)^n}{n} = -\frac{1}{3} \quad \text{and} \quad \sup_{n \geq 3} \frac{(-1)^n}{n} = \frac{1}{4}$$

$\vdots$

$$\inf_{n \geq k} \frac{(-1)^n}{n} = \begin{cases} -\frac{1}{k+1}, & \text{if } k \text{ is even.} \\ -\frac{1}{k}, & \text{if } k \text{ is odd.} \end{cases} \quad \sup_{n \geq k} \frac{(-1)^n}{n} = \begin{cases} \frac{1}{k}, & \text{if } k \text{ is even} \\ \frac{1}{k+1}, & \text{if } k \text{ is odd} \end{cases}$$

If k is even: Window of oscillations is $[-\frac{1}{k+1}, \frac{1}{k}]$

If k is odd: Window of oscillations is $[-\frac{1}{k}, \frac{1}{k+1}]$

$\therefore$ window of oscillations for $\{s_n\} \rightarrow \{0\}$

as $k \rightarrow \infty \Rightarrow$ sequence $\{s_n\}$ converges to 0.

Repeat this for $(-1)^n$:

$$\sup_{n \geq 1} (-1)^n = 1 \quad \text{and} \quad \inf_{n \geq 1} (-1)^n = -1$$

$\Rightarrow$

$$\sup_{n \geq 2} (-1)^n = 1 \quad \text{and} \quad \inf_{n \geq 2} (-1)^n = -1$$

$\vdots$

$\Rightarrow$

$$\sup_{n \geq k} (-1)^n = 1 \quad \text{and} \quad \inf_{n \geq k} (-1)^n = -1$$

$\Rightarrow$ as $k \rightarrow \infty$, the window of oscillations remains $[-1, 1] \Rightarrow (-1)^n$ does not converge as $n \rightarrow \infty$.

To make this precise:

Suppose that we have a sequence $\{s_n\}$. Define

$$u_k = \inf_{n \geq k} s_n, \quad v_k = \sup_{n \geq k} s_n \quad \text{for } k \geq 1$$

$\Rightarrow \{u_k\}$ and $\{v_k\}$ are sequences in $\mathbb{R}$.

Note that

$$\{s_n\}_{n \geq k} \supset \{s_n\}_{n \geq k+1}$$

$$\Rightarrow u_k = \inf_{n \geq k} s_n \leq \inf_{n \geq k+1} s_n = u_{k+1}$$

for any $k \geq 1$ and

$$S \subset \mathbb{R} \text{ and } S_1 \subset S$$

$$\inf S \leq \inf S_1$$

$$\sup S \geq \sup S_1$$

$$v_k = \sup_{n \geq k} s_n \geq \sup_{n \geq k+1} s_n = v_{k+1} \quad \text{for any } k \geq 1$$

$\Rightarrow \{v_k\}$ is monotone $\downarrow$, $\{u_k\}$ is monotone $\uparrow$

Thm
 $\Rightarrow$
 about
 monotone seq.
 both $\{v_k\}$ and $\{u_k\}$ converge as $k \rightarrow \infty$

$$\Rightarrow (*) \lim_{k \rightarrow \infty} u_k \text{ exists} \Leftrightarrow \lim_{k \rightarrow \infty} \left(\inf_{n \geq k} s_n \right) \text{ exists}$$

- denote this by $\liminf_{n \rightarrow \infty} s_n$ or $\underline{\lim}_{n \rightarrow \infty} s_n$
 and call this quantity limit inferior
 of $\{s_n\}$

$$(*) \lim_{k \rightarrow \infty} v_k \text{ exists} \Leftrightarrow \lim_{k \rightarrow \infty} \left(\sup_{n \geq k} s_n \right) \text{ exists}$$

- denote this by $\limsup_{n \rightarrow \infty} s_n$ or $\overline{\lim}_{n \rightarrow \infty} s_n$
 and call this quantity limit superior
 of $\{s_n\}$

Note, $\underline{\lim}_{n \rightarrow \infty} s_n$ and $\overline{\lim}_{n \rightarrow \infty} s_n$ always exist!

$\left(\underline{\lim}_{n \rightarrow \infty} s_n, \overline{\lim}_{n \rightarrow \infty} s_n \right) =$ limiting window of oscillations
 for $\{s_n\}$.

Ex: Find $\overline{\lim}_{n \rightarrow \infty} \frac{n-1}{n+1}$ and $\underline{\lim}_{n \rightarrow \infty} \frac{n-1}{n+1}$.

$$\begin{aligned} (a) \overline{\lim}_{n \rightarrow \infty} \frac{n-1}{n+1} : \quad v_k &= \sup_{n \geq k} \frac{n-1}{n+1} = \sup_{n \geq k} \frac{n+1-2}{n+1} = \sup_{n \geq k} \left(1 - \frac{2}{n+1}\right) \\ &= 1 \Rightarrow \overline{\lim}_{n \rightarrow \infty} \frac{n-1}{n+1} = \lim_{k \rightarrow \infty} \sup_{n \geq k} \frac{n-1}{n+1} \\ &= \lim_{k \rightarrow \infty} v_k = \lim_{k \rightarrow \infty} 1 = 1. \end{aligned}$$

$$\begin{aligned} (b) \underline{\lim}_{n \rightarrow \infty} \frac{n-1}{n+1} : \quad u_k &= \inf_{n \geq k} \frac{n-1}{n+1} = \inf_{n \geq k} \frac{n+1-2}{n+1} = \inf_{n \geq k} \left(1 - \frac{2}{n+1}\right) \\ &= 1 - \frac{2}{k+1} \Rightarrow \underline{\lim}_{n \rightarrow \infty} \frac{n-1}{n+1} = \lim_{k \rightarrow \infty} \inf_{n \geq k} \frac{n-1}{n+1} \\ &= \lim_{k \rightarrow \infty} u_k = \lim_{k \rightarrow \infty} \left(1 - \frac{2}{k+1}\right) = 1. \end{aligned}$$

Ex: Find $\overline{\lim}_{n \rightarrow \infty} (-2)^n$ and $\underline{\lim}_{n \rightarrow \infty} (-2)^n$.

$$\begin{aligned} (a) \overline{\lim}_{n \rightarrow \infty} (-2)^n : \quad v_k &= \sup_{n \geq k} (-2)^n = \infty \Rightarrow \\ \overline{\lim}_{n \rightarrow \infty} (-2)^n &= \lim_{k \rightarrow \infty} \sup_{n \geq k} (-2)^n \\ &= \lim_{k \rightarrow \infty} v_k = \infty \end{aligned}$$

$$\begin{aligned} (b) \underline{\lim}_{n \rightarrow \infty} (-2)^n : \quad u_k &= \inf_{n \geq k} (-2)^n = -\infty \Rightarrow \\ \underline{\lim}_{n \rightarrow \infty} (-2)^n &= \lim_{k \rightarrow \infty} \inf_{n \geq k} (-2)^n \end{aligned}$$

$$= \lim_{k \rightarrow \infty} u_k = -\infty$$

Thm: Suppose that $\lim_{n \rightarrow \infty} s_n = s$, $\lim_{n \rightarrow \infty} t_n = t$
and $s_n \geq t_n, \forall n \in \mathbb{N} \Rightarrow s \geq t$.

Proof: Suppose that, on the contrary $s < t$
 $\Rightarrow$ set $\varepsilon = \frac{t-s}{2} > 0$. By properties of

$$\text{limits, } s - t = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} (s_n - t_n),$$

then, by definition of limit, there

exists a $N \in \mathbb{R}$ s.t. $|(s_n - t_n) - (s - t)| < \frac{t-s}{2}$

$$\text{if } n > N \Rightarrow -\frac{t-s}{2} < \underbrace{(s_n - t_n) - (s - t)} < \frac{t-s}{2}, \text{ if } n > N$$

$$\Rightarrow 0 \leq \underset{\substack{\uparrow \\ \text{nonnegative} \\ \text{because } s_n \geq t_n}}{s_n - t_n} < s - t + \frac{t-s}{2} = \frac{s-t}{2} < 0$$

by our assumption
that $s < t$

- contradiction! $\Rightarrow s \geq t$ $\square$

$$\text{Thm: } \overline{\lim_{n \rightarrow \infty} s_n} \geq \underline{\lim_{n \rightarrow \infty} s_n}$$

Proof: For any $k \in \mathbb{N}$, $u_k = \inf_{n \geq k} s_n \leq \sup_{n \geq k} s_n = v_k$

$\Rightarrow$ By the previous Thm, $\lim_{n \rightarrow \infty} s_n = \lim_{k \rightarrow \infty} u_k \leq \lim_{k \rightarrow \infty} v_k = \overline{\lim_{n \rightarrow \infty} s_n}$ $\square$

Thm: Given a sequence $\{s_n\}$

(1) If $\lim_{n \rightarrow \infty} s_n$ is defined, then $\lim_{n \rightarrow \infty} s_n = \overline{\lim_{n \rightarrow \infty} s_n} = \lim_{n \rightarrow \infty} s_n$

(2) If $\lim_{n \rightarrow \infty} s_n = \overline{\lim_{n \rightarrow \infty} s_n}$, then $\lim_{n \rightarrow \infty} s_n$ is defined and

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_n = \overline{\lim_{n \rightarrow \infty} s_n}.$$

Proof: Will only prove for $\lim_{n \rightarrow \infty} s_n = s_0 \in \mathbb{R}$.

(1) Suppose that $\lim_{n \rightarrow \infty} s_n = s \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{R}$ s.t.

$|s_n - s| < \epsilon$ for $n > N$. Thus,

$$- \epsilon < s_n - s < \epsilon \quad \text{if } n > N,$$

$$s - \epsilon < s_n < s + \epsilon \quad \text{if } n > N,$$

$\Rightarrow$ For any $k > N$, $s + \epsilon$ is an upper bound on

$\{s_n\}_{n=k}^{\infty}$ and $s - \epsilon$ is a lower bound on $\{s_n\}_{n=k}^{\infty}$

$\Rightarrow$ If $k > N$, $\sup_{n \geq k} s_n \leq s + \epsilon$, $\inf_{n \geq k} s_n \geq s - \epsilon$ $\xRightarrow{\text{Thm}}$

$\lim_{k \rightarrow \infty} \sup_{n \geq k} s_n \leq s + \epsilon$ and $\lim_{k \rightarrow \infty} \inf_{n \geq k} s_n \geq s - \epsilon$ $\xRightarrow{\text{Thm}}$

$$s - \epsilon \leq \lim_{n \rightarrow \infty} s_n \leq \overline{\lim_{n \rightarrow \infty} s_n} \leq s + \epsilon.$$

Here ε is arbitrarily small $\Rightarrow$

$$s \leq \lim_{n \rightarrow \infty} s_n \leq \overline{\lim_{n \rightarrow \infty} s_n} \leq s \Rightarrow$$

$$\lim_{n \rightarrow \infty} s_n = \overline{\lim_{n \rightarrow \infty} s_n} = s \quad \square$$

(2) Now suppose that $\lim_{n \rightarrow \infty} s_n = \overline{\lim_{n \rightarrow \infty} s_n} = s$. We want to show that $\lim_{n \rightarrow \infty} s_n = s$. Choose any $\varepsilon > 0$.

Because $\lim_{n \rightarrow \infty} s_n = s$ means that $\lim_{k \rightarrow \infty} u_k = s$, there

exists a $\bar{K}_1 \in \mathbb{R}$ s.t. $|u_k - s| < \varepsilon$ if $k > \bar{K}_1$. Likewise,

because $\overline{\lim_{n \rightarrow \infty} s_n} = s$ means that $\lim_{k \rightarrow \infty} v_k = s$, there

exists a $\bar{K}_2 \in \mathbb{R}$ s.t. $|v_k - s| < \varepsilon$ if $k > \bar{K}_2$.

Suppose now that $k = \max(\bar{K}_1, \bar{K}_2) + 1$.

Then, because $k > \bar{K}_1$, we have $|u_k - s| < \varepsilon$

and because $k > \bar{K}_2$, we have that $|v_k - s| < \varepsilon \Rightarrow$

$$- \varepsilon < u_k - s < \varepsilon \quad \text{and} \quad - \varepsilon < v_k - s < \varepsilon.$$

$\Downarrow$

$\Downarrow$

$$s - \varepsilon < u_k = \inf_{n \geq k} s_n \quad \text{and} \quad \sup_{n \geq k} s_n = v_k < s + \varepsilon$$

$\Downarrow$

$\Downarrow$

$$s - \varepsilon < s_n, \forall n \geq k \quad \text{and} \quad s_n < s + \varepsilon, \forall n \geq k$$

Therefore, if we set $N=k$ and take $n > N \Rightarrow$

$$s - \varepsilon < s_n < s + \varepsilon \Leftrightarrow -\varepsilon < s_n - s < \varepsilon \Leftrightarrow |s_n - s| < \varepsilon.$$

$$\therefore \lim_{n \rightarrow \infty} s_n = s.$$

Recall that $\{s_n\}$ converges if there exists an $s \in \mathbb{R}$ s.t. for $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $|s_n - s| < \varepsilon$

$$\Rightarrow \lim_{n \rightarrow \infty} s_n = s.$$

Def: Suppose that for a sequence $\{s_n\}$, given any $\varepsilon > 0$, there exists an $N \in \mathbb{R}$ s.t. $|s_n - s_m| < \varepsilon$ if $n, m > N$. Then we call the sequence $\{s_n\}$ a Cauchy sequence.

Ex 1.: Show that $s_n = \frac{1}{n}$ is a Cauchy sequence:

Given an $\varepsilon > 0$, want that $|s_n - s_m| < \varepsilon \Rightarrow$

$$|s_n - s_m| = \left| \frac{1}{n} - \frac{1}{m} \right| \leq \left| \frac{1}{n} \right| + \left| \frac{1}{m} \right| = \frac{1}{n} + \frac{1}{m}$$

$$\text{Suppose that } m \leq n \Rightarrow |s_n - s_m| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{m} + \frac{1}{m}$$

$$= \frac{2}{m} < \varepsilon \Leftrightarrow m > \frac{2}{\varepsilon} = N. \text{ Hence, if } n \geq m > N,$$

$$\text{we have } \left| \frac{1}{n} - \frac{1}{m} \right| < \varepsilon \quad \square$$

Thm. 1. Suppose that $\{s_n\}$ converges. Then $\{s_n\}$ is Cauchy

Proof: Since $\{s_n\}$ converges, $\forall \varepsilon > 0$, $\exists N$ s.t.,
 $|s_n - s| < \frac{\varepsilon}{2}$ if $n > N$. Then, suppose that $m, n > N$
and consider $|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|$
 $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$

Thm. 2: Suppose that $\{s_n\}$ is Cauchy. Then $\{s_n\}$ is bounded.

Proof: Suppose that $\{s_n\}$ is Cauchy. Fix any $\varepsilon > 0$.

$\Rightarrow$ def. $\exists N \in \mathbb{R}$ s.t. $|s_n - s_m| < \varepsilon$ if $n, m > N$. Then,
if we choose $m = N+1$, we have $|s_n - s_{N+1}| < \varepsilon$
for any $n > N$. It follows that

$$-\varepsilon < s_n - s_{N+1} < \varepsilon \quad \text{if } n > N$$

$$s_{N+1} - \varepsilon < s_n < s_{N+1} + \varepsilon \quad \text{if } n > N$$

$\Rightarrow s_{N+1} - \varepsilon$ is a lower bound on $\{s_n\}_{n>N}$ and
 $s_{N+1} + \varepsilon$ is an upper bound on $\{s_n\}_{n>N}$

Consider the set $\{s_n\}_{n \geq 1} \Rightarrow \max\{s_1, \dots, s_N, s_{N+1} + \varepsilon\}$
is an upper bound on $\{s_n\}_{n \geq 1}$ and $\min\{s_1, \dots, s_N, s_{N+1} - \varepsilon\}$
is a lower bound on $\{s_n\}_{n \geq 1}$. $\therefore \{s_n\}_{n \geq 1}$ is bounded

from above and below $\Rightarrow \{s_n\}_{n=1}^\infty$ is bounded.

Thm: Suppose that $\{s_n\}$ is a Cauchy sequence.
Then $\{s_n\}$ converges.

Proof: Fix an $\varepsilon > 0$, then there exists a $N \in \mathbb{R}$, s.t.,
 $|s_n - s_m| < \varepsilon$ if $n, m > N \Leftrightarrow -\varepsilon < s_n - s_m < \varepsilon$ if
 $m, n < N \Leftrightarrow s_m - \varepsilon < s_n < s_m + \varepsilon$ if $n, m > N$.

$\Rightarrow$ Fix $m = N+1 \Rightarrow m > N$ and we have that

$$s_{N+1} - \varepsilon < s_n < s_{N+1} + \varepsilon \text{ for all } n > N$$

$\Rightarrow$ Choose any $k > N \Rightarrow s_{N+1} - \varepsilon < s_n < s_{N+1} + \varepsilon$ if

$n \geq k \Rightarrow s_{N+1} + \varepsilon$ is an upper bound on $\{s_n\}_{n \geq k}$

and $s_{N+1} - \varepsilon$ is a lower bound on $\{s_n\}_{n \geq k}$

$$\Rightarrow s_{N+1} + \varepsilon \geq \underbrace{\sup_{n \geq k} s_n}_{V_k} \geq \underbrace{\inf_{n \geq k} s_n}_{U_k} \geq s_{N+1} - \varepsilon, \quad \forall k > N$$

also, $\lim_{k \rightarrow \infty} V_k = \overline{\lim_{n \rightarrow \infty} s_n}$; $\lim_{k \rightarrow \infty} U_k = \underline{\lim_{n \rightarrow \infty} s_n}$, then

sending $k \rightarrow \infty$:

$$s_{N+1} + \varepsilon \geq \overline{\lim_{n \rightarrow \infty} s_n} \geq \underline{\lim_{n \rightarrow \infty} s_n} \geq s_{N+1} - \varepsilon$$

Now letting $m = N+2, N+3, \dots$ we obtain that

$$s_m + \varepsilon \geq \overline{\lim_{n \rightarrow \infty}} s_n \geq \underline{\lim_{n \rightarrow \infty}} s_n \geq s_m - \varepsilon \text{ for any } m > N.$$

$\Downarrow$

$$s_m \geq \overline{\lim_{n \rightarrow \infty}} s_n - \varepsilon \text{ for any } m > N \Rightarrow$$

If $k > N$ and $m \geq k$ we have $s_m \geq \overline{\lim_{n \rightarrow \infty}} s_n - \varepsilon$

$\Rightarrow \overline{\lim_{n \rightarrow \infty}} s_n - \varepsilon$ is a lower bound on $\{s_m\}_{m \geq k}$

$$\Rightarrow \inf_{m \geq k} s_m \geq \overline{\lim_{n \rightarrow \infty}} s_n - \varepsilon \text{ for any } k > N.$$

$\downarrow k$

$\downarrow n \rightarrow \infty$

$$\underline{\lim_{n \rightarrow \infty}} s_n \geq \overline{\lim_{n \rightarrow \infty}} s_n - \varepsilon, \quad \forall \varepsilon > 0$$

Letting $\varepsilon \rightarrow 0$: $\underline{\lim_{n \rightarrow \infty}} s_n \geq \overline{\lim_{n \rightarrow \infty}} s_n$. Since always

$$\overline{\lim_{n \rightarrow \infty}} s_n \geq \underline{\lim_{n \rightarrow \infty}} s_n \Rightarrow \overline{\lim_{n \rightarrow \infty}} s_n = \underline{\lim_{n \rightarrow \infty}} s_n \stackrel{\text{Thm}}{\Rightarrow}$$

$\lim_{n \rightarrow \infty} s_n$ exists. $\square$

Therefore.

Thm. Any sequence of real numbers converges if and only if it is Cauchy.

Ex. Suppose that $s_n = 0.123\dots n \Rightarrow \{s_n\} \subset \mathbb{Q} \subset \mathbb{R}$

In $\mathbb{R}$ this sequence converges to $0.1234\dots$ - nonterminating, nonrepeating decimal $\Rightarrow \{s_n\}$ converges to an irrational # in $\mathbb{R} \xRightarrow{\text{Thm}} \{s_n\}$ is a Cauchy sequence in $\mathbb{R} \Rightarrow \{s_n\}$ is a Cauchy sequence in $\mathbb{Q}$. However, the limit of $\{s_n\}$ is not in $\mathbb{Q}$. Therefore, Cauchy sequences in $\mathbb{Q}$ do not have to be convergent in $\mathbb{Q}$.