Consider a sequence $s_n = \frac{1}{n}$:

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Can also consider $t_n = \frac{1}{2n}$:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \ldots$ – all elements of $\{t_n\}$ are also elements of $\{s_n\}$ and appear in $\{s_n\}$ in the same order

as well as a sequence $p_n = \frac{1}{3n+1}$:

$\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \ldots$ – all elements of $\{t_n\}$ are also elements of $\{s_n\}$ and appear in $\{s_n\}$ in the same order

$\Rightarrow \{p_n\}$ and $\{t_n\}$ are "subsequences" of $\{s_n\}$.

To relate indices of $\{s_n\}$ and $\{t_n\}$:

(1) Relabel $\{t_n\}$: $\{t_k\}_{k \geq 1}$ so that $t_k = \frac{1}{2k} \Leftrightarrow$

$t(k) = \frac{1}{2k}$

(2) $s_n = \frac{1}{n} \Leftrightarrow s(n) = \frac{1}{n}$

$$\left.\begin{array}{l} t(k) = \frac{1}{2k} \\ s_n = \frac{1}{n} \Leftrightarrow s(n) = \frac{1}{n} \end{array}\right\} t_k = \frac{1}{2k} \overset{n=2k}{=} s(2k) = s_{2k}$$

To relate indices of $\{s_n\}$ and $\{p_n\}$:

(1) Relabel $\{p_n\}$: $\{p_k\}_{k \geq 1}$ so that $p_k = \frac{1}{3k+1} \Leftrightarrow$

$p(k) = \frac{1}{3k+1}$

(2) $s_n = \frac{1}{n} \Leftrightarrow s(n) = \frac{1}{n}$

$$\left.\begin{array}{l} p(k) = \frac{1}{3k+1} \\ s_n = \frac{1}{n} \Leftrightarrow s(n) = \frac{1}{n} \end{array}\right\} p_k = \frac{1}{3k+1} \overset{n=3k+1}{=} s(3k+1) = s_{3k+1}$$

Def: Given a sequence $\{s_n\}$ and a positive, strictly increasing, integer-valued function $n(k)$, a subsequence $\{s_{n_k}\}$ is defined as $s_{n_k} = s(n(k))$.

Ex: $s_n = (-1)^n \Rightarrow -1, 1, -1, 1, -1, \ldots$

(*) $s_{2k} = (-1)^{2k} = 1 \Rightarrow 1, 1, 1, 1, \ldots$ – a subsequence of $\{s_n\}$

(*) $s_{2k+1} = (-1)^{2k+1} = -1 \Rightarrow -1, -1, -1, -1, \ldots$ – a subsequence of $\{s_n\}$

(*) $s_{4k} = (-1)^{4k} = 1 \Rightarrow 1, 1, 1, 1, \ldots$ – a subsequence of $\{s_n\}$

(*) $-1, -1, 1, 1, -1, -1, 1, 1, \ldots$ – a subsequence of $\{s_n\}$

Ex: $s_n = \frac{1}{2^n}$. Question:

(1) Is $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{8}, \frac{1}{32}, \ldots$ a subsequence of $\{s_n\}$?

$$\begin{array}{cccccc} \frac{1}{2}, & \frac{1}{4}, & \frac{1}{16}, & \frac{1}{8}, & \frac{1}{32}, & \frac{1}{64}, \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ n=1 & n=2 & n=4 & n=3 & n=5 & n=6 \end{array}$$

$$n(1)=1;\ n(2)=2;\ \underbrace{n(3)=4;\ n(4)=3}_{n(3)>n(4)\ \Rightarrow\ n(k) \text{ is not } \uparrow};\ n(5)=5;\ n(6)=6$$

$\Rightarrow$ not a subsequence.

(2) $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots$
s_2, s_4, s_6, s_8 $\Bigg\} \Rightarrow$ this is a subsequence $\{s_{2k}\}$

<u>Thm:</u> suppose that $n(k)$ is a strictly $\uparrow$ function from positive integers to positive integers. Then $n(k) \geq k$ for every $k \geq 1$.

<u>Proof:</u> Use induction. Because 1 is the smallest positive integer, we have that $n(1) \geq 1$. Now suppose that $n(k) \geq k \Rightarrow n(k+1) > n(k)$
$\Rightarrow n(k+1) \geq n(k)+1 \geq k+1$ $\boxtimes$

<u>Thm:</u> Suppose $\{s_n\}$ is unbounded from above (or below)
$\Rightarrow$ there exists an $\uparrow(\downarrow)$ subsequence $\{s_{n_k}\}$ s.t. $\lim_{k\to\infty} s_{n_k} = \infty$ (or $-\infty$).

<u>Proof:</u> Suppose $\{s_n\}$ is not bounded from above. Let $n_1 = 1$ and consider the number $\max\{2, s_{n_1}\} \in \mathbb{R}$. This number is not an upper bound on $\{s_n\}$ because $\{s_n\}$ is not bounded from above
$\Rightarrow$ there exists $n_2 > 1$ s.t. $s_{n_2} > \max\{2, s_{n_1}\} \Leftrightarrow s_{n_2} > 2$ and $s_{n_2} > s_{n_1}$
Use induction: Suppose $s_{n_k} > s_{n_{k-1}}$ and $s_{n_k} > k$. Because $\max\{s_{n_k}, k+1\}$ is not an upper bound on $\{s_n\}$, there exists $n_{k+1} \in \mathbb{N}$ s.t. $s_{n_{k+1}} > \max\{s_{n_k}, k+1\} \Leftrightarrow$

$s_{n_{k+1}} > s_{n_k}$ and $s_{n_{k+1}} > k+1$

$\downarrow$ $\qquad\qquad$ $\downarrow$

s_{n_k} is $\uparrow$ $\qquad\qquad$ $\lim_{k\to\infty} s_{n_{k+1}} \geq \lim_{k\to\infty}(k+1) = \infty$

$\boxtimes$

Thm: Given a sequence $\{s_n\}$, it contains a subsequence $\{s_{n_k}\}$ convergent to some $t \in \mathbb{R}$ iff the set $\{s_n : |s_n - t| < \varepsilon\}$ contains ∞ many elements for any $\varepsilon > 0$.

Proof: (1) Suppose that there is an $\{s_{n_k}\}$ s.t. $\lim_{k\to\infty} s_{n_k} = t$

$\Rightarrow$ $\forall \varepsilon > 0$, there exists a $\overline{K} \in \mathbb{R}$ s.t. $|s_{n_k} - t| < \varepsilon$ for $\forall k > \overline{K}$

but there are ∞ many k s.t. $k > \overline{K}$ $\Rightarrow$ $\{s_n : |s_n - t| < \varepsilon\}$

has ∞ many elements. $\boxtimes$

(2) Now, suppose that the set $\{s_n : |s_n - t| < \varepsilon\}$ contains ∞ many elements for any $\varepsilon > 0$ $\Leftrightarrow$ the set

$\{s_n : -\varepsilon < s_n - t < \varepsilon\}$ contains ∞ many elements $\forall \varepsilon > 0$ $\Leftrightarrow$

the set $\{s_n : t - \varepsilon < s_n < t\} \cup \{s_n : t < s_n < t + \varepsilon\}$ contains ∞ many elements $\forall \varepsilon > 0$ $\Leftrightarrow$ either $\{s_n : t - \varepsilon < s_n < t\}$ contains ∞ many elements $\forall \varepsilon > 0$ OR $\{s_n : t < s_n < t + \varepsilon\}$ contains ∞ many elements $\forall \varepsilon > 0$.

W.l.o.g., assume $\{s_n : t - \varepsilon < s_n < t\}$ contains ∞ many elements $\forall \varepsilon > 0$.

$\Rightarrow$ Let $\varepsilon_1 = 1$. Because the set $\{t - \varepsilon_1 < s_n < t\}$ contains ∞ many elements, we can choose $n_1 \in \mathbb{N}$ s.t. $t - \varepsilon_1 < s_{n_1} < t$. Next, let $\varepsilon_2 = \min\{t - s_{n_1}, \frac{1}{2}\}$. Because the set $\{t - \varepsilon_2 < s_n < t\}$ contains ∞ many elements, we can choose $n_2 \in \mathbb{N}$ s.t.

$t - \varepsilon_2 < s_{n_2} < t$ $\Leftrightarrow$ $t - \min\{t - s_{n_1}, \frac{1}{2}\} < s_{n_2} < t$ $\Leftrightarrow$

$$t - \frac{1}{2} < s_{n_2} < t \quad \underline{\text{and}} \quad s_{n_1} < s_{n_2} < t$$

Continue by induction: Suppose that

$$t-\frac{1}{k}<s_{n_k}<t \quad \text{and} \quad s_{n_{k-1}}<s_{n_k}<t$$

Let $\varepsilon_{k+1}=\min\{t-s_{n_k}, \frac{1}{k+1}\}$. Because the set $\{t-\varepsilon_{k+1}<s_n<t\}$ contains ∞ many elements, we can choose $n_{k+1}\in\mathbb{N}$ s.t. $t-\varepsilon_{k+1}<s_{n_{k+1}}<t$

$$\iff \quad t-\min\{t-s_{n_k}, \tfrac{1}{k+1}\}<s_{n_{k+1}}<t \quad \iff$$

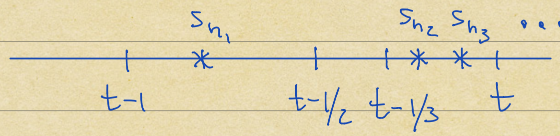
$$t-\frac{1}{k+1}<s_{n_{k+1}}<t \quad \underline{\text{and}} \quad s_{n_k}<s_{n_{k+1}}<t$$

Therefore, we constructed a subsequence $\{s_{n_k}\}$ s.t.:

(a) $t-\frac{1}{k}<s_{n_k}<t$ (b) $s_{n_k}<s_{n_{k+1}}, \forall k\in\mathbb{N} \Rightarrow \{s_{n_k}\}$ is monotone $\uparrow$.

$k\to\infty$: $t-\frac{1}{k}\to t$, $t \to t$

$\Rightarrow \lim\limits_{k\to\infty} s_{n_k}=t$ ⊠



Ex. Question: Is it possible to construct a sequence of all positive rational numbers listed in order of magnitude? Clearly, this is possible for natural #'s: just set $s_n=n$. Suppose that the same can be done for rationals: for any $q\in\mathbb{Q}_+$ - set of all positive rationals, we have that $s_{n_0}=q$ for some $n_0\in\mathbb{N}$; also if $q_0>q_1 \Rightarrow s_{n_0}=q_0, s_{n_1}=q_1$, where $n_0>n_1$. Then q_1 should be the smalles positive rational # $\Rightarrow$ by denseness of $\mathbb{Q}$, there exists an $s\in\mathbb{Q}$ s.t. $s\in(0,q_1) \Rightarrow s$ is a positive rational number $< q_1 \Rightarrow s$ must appear in $\{s_n\}$ before q_1 – contradiction.

$\therefore$ Positive rational numbers cannot be listed as a sequence in increasing order.

Next, let the height of rational # $\frac{p}{q}$ to be $p+q$ when the number is written in the irreducible form. E.g.,

$$\text{Height of } \frac{1}{2} = 1+2 = 3$$

$$\text{Height of } \frac{3}{9} = \text{Height of } \frac{1}{3} = 1+3 = 4$$

etc.

Construct a table of positive heights:

Height	All rationals w/given height:
1	–
2	$1 \leftarrow s_1$
3	$s_2 \rightarrow 1/2$, $2 \leftarrow s_3$
4	$s_4 \rightarrow 1/3$, $3 \leftarrow s_5$
5	$1/4$, $2/3$, $3/2$, 4
⋮	s_6 s_7 s_8 s_9 …

– construct a sequence that lists <u>all</u> positive rationals.

<u>Note:</u> (1) Every rational # has a height $\Rightarrow$ every rational # will appear on some line in this table

(2) There are finitely many positive rationals

with a given height.

Important: Rationals in this sequence are not is an increasing order!

Next, let t be any positive real #. Question: How many rational #'s are there in the interval $(t-\varepsilon, t+\varepsilon)$ for any $\varepsilon > 0$? By denseness of $\mathbb{Q}$ there are ∞ many rationals like that. Therefore, for any $\varepsilon > 0$, there ∞ many terms of the sequence $\{s_n\}$ of all positive rationals inside $(t-\varepsilon, t+\varepsilon)$.
$\overset{\text{Thm}}{\Longrightarrow}$ there exists a subsequence $\{s_{n_k}\}$ s.t. $\lim\limits_{k\to\infty} s_{n_k} = t$.

Thm: Suppose that $\lim\limits_{n\to\infty} s_n = s$. Then any subsequence of $\{s_n\}$ converges to s.

Proof: Because $\lim\limits_{n\to\infty} s_n = s$, by def. of the limit, for any $\varepsilon > 0$, there exists an $N \in \mathbb{R}$ s.t. $n > N \Rightarrow$ $|s_n - s| < \varepsilon$. Consider any subsequence $\{s_{n_k}\}$ of $\{s_n\}$ and let $k > N$. We proved that $n_k \geq k \Rightarrow$ $n_k \geq k > N \Rightarrow |s_{n_k} - s| < \varepsilon \Rightarrow \lim\limits_{k\to\infty} s_{n_k} = s$ ⊠

<u>Thm</u>: Given any sequence $\{s_n\}$, the sequence always contains a monotone subsequence.

<u>Proof</u>: (1) We will say that s_{n_0} is dominant if $s_{n_0} \geq s_n$ for any $n > n_0$. Assume that $\{s_n\}$ contains ∞ many dominant terms. Consider a subsequence that consists of all dominant terms: $\{s_{n_k}\}$ has a property $s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq s_{n_4} \cdots \geq s_{n_{k-1}} \geq s_{n_k} \geq$

$\therefore$ the subsequence of dominant term is $\downarrow$.

(2) Now suppose that $\{s_n\}$ has finitely many dominant terms $\Rightarrow$ there exists an index n_1 s.t. there are no dominant terms when $n \geq n_1$ $\Rightarrow$ s_{n_1} is not dominant $\Rightarrow$ there exists $n_2 > n_1$ s.t. $s_{n_2} > s_{n_1}$; s_{n_2} is not dominant $\Rightarrow$ there exists $n_3 > n_2$ s.t. $s_{n_3} > s_{n_2}$ $\Rightarrow$ by induction we construct $\{s_{n_k}\}$ s.t. s_{n_k} is $\uparrow$. $\boxtimes$

Bolzano-Weierstrass Thm: Given any bounded sequence $\{s_n\}$, there exists a convergent subsequence $\{s_{n_k}\}$.

<u>Proof</u>: We already showed that any sequence has a monotonic subsequence $\Rightarrow$

there exists a monotonic subsequence $\{s_{n_k}\}$. Further, $\{s_n\}$ is bounded $\Rightarrow$ $\{s_{n_k}\}$ is also bounded $\Rightarrow$ $\{s_{n_k}\}$ is a monotone bounded sequence $\Rightarrow$ $\{s_{n_k}\}$ converges $\boxtimes$

Ex: $s_n = (-1)^n$ – does not converge; What are possible limits of convergent subsequences of $\{s_n\}$? Ans: $+1$ and -1.

Set of subsequential limits: $S = \{-1, 1\}$

Ex: $s_n = \frac{n+1}{2n}$; What are possible limits of subsequences?

$$\lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2} \Rightarrow$$

any subsequence of $\{s_n\}$ converge to $\frac{1}{2}$.

Set of subsequential limits: $S = \{\frac{1}{2}\}$

Ex: $s_n = (-n)^n$ What are possible limits of subsequences?

Here $\lim_{n\to\infty} s_n = \lim_{n\to\infty} (-n)^n = \lim_{n\to\infty} (-1)^n n^n$

$\Rightarrow$ $\{s_n\}$ is not bounded from above $\Rightarrow$ there is a subsequence of $\{s_n\}$ converging to ∞.

$\{s_n\}$ is not bounded from below $\Rightarrow$
there is a subsequence of $\{s_n\}$ converging to $-\infty$.

set of subsequential limits: $S = \{-\infty, \infty\}$

Ex. Consider a sequence of all positive rationals. We showed that this sequence has subsequences that converge to all positive real numbers.

set of subsequential limits: $S = \{x \in \mathbb{R} : x \geq 0\}$

<u>Def</u>: S is the set of all subsequential limits of $\{s_n\}$ if $s \in S \iff$ there exists a subsequence of $\{s_n\}$ that converges to s.

<u>Thm 1</u>: Given any $\{s_n\}$ in $\mathbb{R}$, denote by S the set of all subsequential limits of $\{s_n\}$.

(1) S is not empty.

(2) $\sup S = \overline{\lim_{n\to\infty}} s_n$, $\inf S = \underline{\lim_{n\to\infty}} s_n$

(3) $\{s_n\}$ converges iff S contains a single element.

Thm 2: Given a sequence $\{s_n\}$ there exists a monotone subsequence convergent to $\overline{\lim\limits_{n\to\infty}} s_n$ and a monotone subsequence convergent to $\underline{\lim\limits_{n\to\infty}} s_n$.

Proof: (1) Suppose that $\{s_n\}$ is not bounded from above $\Rightarrow$ $\overline{\lim\limits_{n\to\infty}} s_n = \infty$ and we already proved that there is a subsequence of $\{s_n\}$ that converges to ∞ $\Rightarrow$ $\infty \in S$ $\boxtimes$

(2) Suppose that $\{s_n\}$ is not bounded from below - same argument as in (1).

(3) W.l.o.g. suppose that $\{s_n\}$ is bounded from above $\Rightarrow$ $\overline{\lim\limits_{n\to\infty}} s_n$ is finite $\Rightarrow$

if $t = \overline{\lim\limits_{n\to\infty}} s_n \iff t = \lim\limits_{k\to\infty} \sup\limits_{n\geq k} s_n \Rightarrow$

for any $\varepsilon > 0$, $\exists N$ s.t. $k > N \Rightarrow |\sup\limits_{n\geq k} s_n - t| < \varepsilon$

$\iff k > N \Rightarrow t - \varepsilon < \sup\limits_{n\geq k} s_n < t + \varepsilon$. Therefore,

$\sup\limits_{n\geq k} s_n < t + \varepsilon$ if $k > N \Rightarrow \sup\limits_{n\geq N+1} s_n < t+\varepsilon$

$\iff \sup_{n>N} s_n < t+\varepsilon$. Now consider the interval $(t-\varepsilon, t+\varepsilon)$. How many s_n's are in this interval? Suppose that there are finitely many s_n like this $\Rightarrow$ there exists an $N_1 > N$ s.t. $s_n \notin (t-\varepsilon, t+\varepsilon)$ if $n > N_1 \Rightarrow \sup_{n\geq k} s_n \leq t-\varepsilon$ if $k > N_1 \Rightarrow \lim_{k\to\infty} \sup_{n\geq k} s_n \leq t-\varepsilon \iff \overline{\lim_{n\to\infty}} s_n \leq t-\varepsilon$ – contradiction $\Rightarrow (t-\varepsilon, t+\varepsilon)$ contains ∞ many $s_n \overset{\text{Thm}}{\Rightarrow}$ there exists a mon. subsequence of $\{s_n\}$ that converges to t.

<u>Proof</u> of Thm 1: (1) follows from Thm 2., just take, e.g. a subsequence $\to \overline{\lim_{n\to\infty}} s_n$.

(2) Suppose that $t \in S \Rightarrow$ there exists a subsequence $\{s_{n_k}\}$ s.t. $\lim_{k\to\infty} s_{n_k} = t$.

$\inf_{l\geq k} s_l \leq s_{n_k} \leq \sup_{l\geq k} s_l$ for any k, since $s_{n_k} \in \{s_l, l \geq k\}$

$$\Rightarrow \quad \lim_{k\to\infty} \inf_{l \geq k} s_l \leq \lim_{k\to\infty} s_{n_k} \leq \lim_{k\to\infty} \sup_{l\geq k} s_l$$

$$\Rightarrow \quad \underline{\lim}_{n\to\infty} s_n \leq t \leq \overline{\lim}_{n\to\infty} s_n \quad \boxtimes$$

(3) $(\Rightarrow)$ $S = \{s\} \overset{\text{Thm2}}{\Rightarrow} \overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{n\to\infty} s_n = s \overset{\text{Thm}}{\Rightarrow}$

$$\lim_{n\to\infty} s_n = s.$$

$(\Leftarrow)$ If $\lim_{n\to\infty} s_n = s \overset{\text{Thm}}{\Rightarrow} \lim_{k\to\infty} s_{n_k} = s$ for any

$\{s_{n_k}\} \Rightarrow S = \{s\}.$

$$\begin{pmatrix} 1/3 & & 0 \\ & \frac{1}{3} & \\ 0 & & \frac{1}{3} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{3}{2} \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{pmatrix}$$