Thm. 1: Suppose $\lim_{n\to\infty} s_n = s > 0$ and $\{t_n\}$ is any sequence. Then

$$\overline{\lim_{n\to\infty}}\, s_n t_n = s\, \overline{\lim_{n\to\infty}}\, t_n.$$

Proof: (1) Suppose $\overline{\lim_{n\to\infty}}\, t_n = t$ - finite.

(a) Show that $\overline{\lim_{n\to\infty}}\, s_n t_n \geq s\, \overline{\lim_{n\to\infty}}\, t_n$

By a Thm, we can extract $\{t_{n_k}\}$ s.t. $\lim_{k\to\infty} t_{n_k} = t$. If we consider the subsequence $\{s_{n_k}\}$, then $\lim_{k\to\infty} s_{n_k} = s$, because any subsequence of a convergent sequence has the same limit as the original sequence $\Rightarrow \lim_{k\to\infty} s_{n_k} t_{n_k} = st$ by properties of limits. We then conclude that st belongs to the set of subsequential limits of $\{s_n t_n\}$. By a theorem,

$$\overline{\lim_{n\to\infty}}\, s_n t_n = \sup S$$ - set of all subsequential limits of $\{s_n t_n\}$

$\therefore \quad st \leq \overline{\lim_{n\to\infty}}\, s_n t_n$

$$\Rightarrow s\, \overline{\lim_{n\to\infty}}\, t_n \leq \overline{\lim_{n\to\infty}}\, s_n t_n$$

(b) Show that

$$\overline{\lim_{n\to\infty}}\, s_n t_n \leq s\, \overline{\lim_{n\to\infty}}\, t_n$$

Because $\lim_{n\to\infty} s_n = s > 0$, starting from some n, all

$s_n > 0 \Rightarrow \lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s} \Rightarrow$

$$\overline{\lim_{n\to\infty}}\, t_n = \overline{\lim_{n\to\infty}}\, \frac{1}{s_n}(s_n t_n) \geq \frac{1}{s} \overline{\lim_{n\to\infty}}\, s_n t_n \stackrel{s>0}{\Longrightarrow}$$

$$s \overline{\lim_{n\to\infty}}\, t_n \geq \overline{\lim_{n\to\infty}}\, s_n t_n \text{ - done.}$$

(2) Let $\overline{\lim_{n\to\infty}}\, t_n = \infty \stackrel{\text{Thm}}{\Longrightarrow} \exists \{t_{n_k}\}$ s.t. $\lim_{k\to\infty} t_{n_k} = \infty$

Then, $\lim_{k\to\infty} s_{n_k} = s > 0 \underset{\text{lims}}{\overset{\text{Prop of}}{\Longrightarrow}} \lim_{k\to\infty} s_{n_k} t_{n_k} = \infty \Rightarrow$

supremum of the set of subsequential limits $= \infty \Rightarrow$

$$\overline{\lim_{n\to\infty}}\, s_n t_n = \infty.$$

Because "$s \cdot \infty = \infty$", $s \overline{\lim_{n\to\infty}}\, t_n = \overline{\lim_{n\to\infty}}\, s_n t_n$ - done

(3) If we let $\overline{\lim_{n\to\infty}}\, t_n = -\infty \Rightarrow s \overline{\lim_{n\to\infty}}\, t_n = -\infty \Rightarrow$

$$\overline{\lim_{n\to\infty}}\, s_n t_n \geq s \overline{\lim_{n\to\infty}}\, t_n$$

because any real #, $\pm\infty \geq -\infty$. Now use 1(b) to finish the proof.

<u>Thm 2</u>: For any sequence $\{s_n\}$ of real #'s:

$$\underline{\lim_{n\to\infty}} \left|\frac{s_{n+1}}{s_n}\right| \leq \underline{\lim_{n\to\infty}} \sqrt[n]{|s_n|} \leq \overline{\lim_{n\to\infty}} \sqrt[n]{|s_n|} \leq \overline{\lim_{n\to\infty}} \left|\frac{s_{n+1}}{s_n}\right|$$

<u>Proof:</u> Middle inequality is obvious; if we can prove the right inequality

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|s_n|} \le \overline{\lim_{n\to\infty}}\left|\frac{s_{n+1}}{s_n}\right|, \qquad (*)$$

then the left inequality can be proved in the same way.

(1) If $\overline{\lim\limits_{n\to\infty}}\left|\frac{s_{n+1}}{s_n}\right| = \infty$, then $(*)$ is always true.

(2) Suppose that $\overline{\lim\limits_{n\to\infty}}\left|\frac{s_{n+1}}{s_n}\right| = L$ - finite nonnegative number. We want to show that

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|s_n|} \le L.$$

This is equivalent to proving that

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|s_n|} \le L_1, \ \forall L_1 > L.$$

Choose such L_1, then

$$L_1 > L = \overline{\lim_{n\to\infty}}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{k\to\infty}\sup_{n\ge k}\left|\frac{s_{n+1}}{s_n}\right| \Rightarrow \exists N\in\mathbb{R}, \text{s.t.,}$$

$$\sup_{n\ge k}\left|\frac{s_{n+1}}{s_n}\right| < L_1 \text{ if } k > N \Rightarrow \sup_{n>N}\left|\frac{s_{n+1}}{s_n}\right| < L_1$$

which is equivalent to $\left|\frac{s_{n+1}}{s_n}\right| < L_1$ for all $n > N \Rightarrow$

$\left|\frac{s_{N+2}}{s_{N+1}}\right| < L_1$; $\left|\frac{s_{N+3}}{s_{N+2}}\right| < L_1$, etc. $\Rightarrow$ for any $n > N$, we have

$$\left|\frac{s_n}{s_{N+1}}\right| = \left|\frac{s_n}{s_{n-1}}\cdot\frac{s_{n-1}}{s_{n-2}}\cdot\ldots\cdot\frac{s_{N+3}}{s_{N+2}}\cdot\frac{s_{N+2}}{s_{N+1}}\right|$$

$$= \left|\frac{s_n}{s_{n-1}}\right|\cdot\left|\frac{s_{n-1}}{s_{n-2}}\right|\cdot\ldots\cdot\left|\frac{s_{N+3}}{s_{N+2}}\right|\cdot\left|\frac{s_{N+2}}{s_{N+1}}\right| < \overbrace{L_1\cdot L_1\cdot\ldots\cdot L_1}^{n-N-1 \text{ times}} = L_1^{\,n-N-1} = \frac{L_1^n}{L_1^{N+1}}$$

$\Rightarrow \quad |s_n| < \dfrac{|s_{N+1}|}{L_1^{N+1}} L_1^n, \ \forall n > N \Rightarrow \sqrt[n]{|s_n|} < \left(\dfrac{|s_{N+1}|}{L_1^{N+1}}\right)^{1/n} L_1, \forall n > N$

$\Rightarrow \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|s_n|} \le \overline{\lim_{n\to\infty}} \Big(\underbrace{\dfrac{|s_{N+1}|}{L_1^{N+1}}}_{=\text{const}}\Big)^{1/n} L_1 = \lim_{n\to\infty} \left(\dfrac{|s_{N+1}|}{L_1^{N+1}}\right)^{1/n} L_1$

$\underset{\text{of lim}}{\overset{\text{Prop}}{=}} L_1 \lim_{n\to\infty} \left(\dfrac{|s_{N+1}|}{L_1^{N+1}}\right)^{1/n} \underset{\lim\limits_{n\to\infty} a^{\frac{1}{n}} = 1}{=} L_1 \quad \Rightarrow \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|s_n|} < L_1, \forall L_1 > L$

$\Rightarrow \overline{\lim_{n\to\infty}} \sqrt[n]{|s_n|} \le L = \overline{\lim_{n\to\infty}} \left|\dfrac{s_{n+1}}{s_n}\right| \quad \blacksquare$