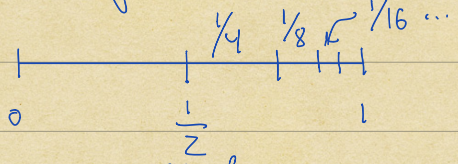
Given a sequence $\{s_n\}$ a series associated with $\{s_n\}$ is defined as:

$$\sum_{n=1}^{\infty} s_n = \lim_{k\to\infty} \underbrace{\sum_{n=1}^{k} s_n}$$

↳ k-th partial sum of the series.

If the limit exists ⇒ series converges; if the limit is $\pm\infty$, the series diverges, otherwise the series does not converge.

(1) 

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1$$

(2) $$\underbrace{\sum_{n=0}^{\infty} q^n}_{\leftarrow \text{ geometric series}} = 1 + q + q^2 + \dots + q^n + \dots = \lim_{k\to\infty}\sum_{n=0}^{k} q^n = \lim_{k\to\infty}\frac{1-q^{k+1}}{1-q}$$

multiplier.

Consider a partial sum $S_k = \sum_{n=0}^{k} q^n = 1 + q + q^2 + \dots + q^k$

$$\Rightarrow qS_k = q(1+q+\dots+q^k) = q + q^2 + \dots + q^k + q^{k+1} = S_k - 1 + q^{k+1}$$

$$qS_k = S_k - 1 + q^{k+1} \Rightarrow 1 - q^{k+1} = S_k - qS_k = (1-q)S_k$$

$$\Rightarrow S_k = \frac{1-q^{k+1}}{1-q} \Rightarrow$$

$$\sum_{n=0}^{\infty} q^n = \lim_{k\to\infty}\frac{1-q^{k+1}}{1-q}$$

$$= \begin{cases} \frac{1}{1-q}, & \text{if } |q|<1 \\ \infty, & \text{if } q \geq 1 \\ \text{d.n.e}, & \text{if } q \leq -1 \end{cases}$$

$$\boxed{\lim_{k\to\infty} q^k = \begin{cases} 0, & \text{if } |q|<1 \\ 1, & \text{if } q=1 \\ \text{d.n.e.}, & \text{if } q \leq -1 \\ \infty, & \text{if } q > 1 \end{cases}}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} - 1 = 1$$

(3) $\displaystyle\sum_{n=2}^{\infty} \frac{5^{n-1}\cdot 3^{n+2}}{2^{4n}}$ – does this series converge? If yes, what is the sum of the series?

$$\frac{5^{n-1}\cdot 3^{n+2}}{2^{4n}} = \frac{5^n\cdot 5^{-1}\cdot 3^n\cdot 3^2}{(2^4)^n} = \frac{9}{5}\,\frac{5^n\cdot 3^n}{16^n} = \frac{9}{5}\left(\frac{15}{16}\right)^n$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{5^{n-1}\cdot 3^{n+2}}{2^{4n}} = \sum_{n=2}^{\infty}\frac{9}{5}\left(\frac{15}{16}\right)^n = \frac{9}{5}\underbrace{\sum_{n=2}^{\infty}\left(\frac{15}{16}\right)^n}_{\substack{\text{geometric series}\\ \text{w/ } q=\frac{15}{16}<1}}$$

$\Rightarrow$ series converges.

$$\frac{9}{5}\sum_{n=2}^{\infty}\left(\frac{15}{16}\right)^n = \frac{9}{5}\left(\sum_{n=0}^{\infty}\left(\frac{15}{16}\right)^n - 1 - \frac{15}{16}\right)$$

$$= \frac{9}{5}\left(\frac{1}{1-\frac{15}{16}} - 1 - \frac{15}{16}\right) = \frac{9}{5}\left(16 - 1 - \frac{15}{16}\right)$$

(4) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \Rightarrow$

$S_0 = (-1)^0 = 1$

$S_1 = (-1)^0 + (-1)^1 = 0$

$\lim_{n\to\infty} S_k$ d.n.e $\Leftarrow$ $S_2 = (-1)^0 + (-1)^1 + (-1)^2 = 1$

$S_3 = (-1)^0 + (-1) + (-1)^2 + (-1)^3 = 0$

– the series does not converge.

(5) Harmonic series: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

Observe: $1 > \frac{1}{2}$; $\frac{1}{2} \geq \frac{1}{2}$; $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$;

$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$; $\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}$

$+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16} > 8\cdot\frac{1}{16} = \frac{1}{2}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} > \sum_{n=1}^{\infty} \frac{1}{2} = \infty$$

$\hookrightarrow$ diverge.

Recall, that $\sum_{n=0}^{\infty} a_n$ converges if $\lim_{k\to\infty} S_k$ exists,

where $S_k = \sum_{n=0}^{k} a_n \underset{\substack{a_n \in \mathbb{R} \\ \forall n \geq 0}}{\Longleftrightarrow} \{S_k\}$ is a Cauchy sequence

$\Longleftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{R}$, s.t. $l \geq m > N \Rightarrow |S_l - S_m| < \varepsilon$

$\Longleftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{R}$, s.t. $l \geq m > N \Rightarrow \left|\sum_{n=0}^{l} a_n - \sum_{n=0}^{m} a_n\right| < \varepsilon$

$\Longleftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{R}$, s.t. $l > m > N \Rightarrow \left|\sum_{n=m+1}^{l} a_n\right| < \varepsilon$

$\overset{p=m+1}{\Longleftrightarrow} \forall \varepsilon > 0, \exists N \in \mathbb{R}$, s.t. $l \geq p > N \Rightarrow \left|\sum_{n=p}^{l} a_n\right| < \varepsilon$

$\therefore$ If a series $\sum_{n=0}^{\infty} a_n$ satisfies Cauchy criterion:

$\forall \varepsilon > 0, \exists N \in \mathbb{R}$, s.t. $l \geq p > N \Rightarrow \left|\sum_{n=p}^{l} a_n\right| < \varepsilon$

$\Rightarrow \sum_{n=0}^{\infty} a_n$ converges.

Thm: (Test for divergence) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ does not converge.

Proof: Suppose that, on the contrary $\sum_{n=0}^{\infty} a_n$ converges

$\Rightarrow \sum_{n=0}^{\infty} a_n$ satisfies the Cauchy criterion:

$$\forall \varepsilon > 0, \exists N \in \mathbb{R}, \text{ s.t. } l \geq p > N \Rightarrow \left|\sum_{n=p}^{l} a_n\right| < \varepsilon$$

Take $l = p > N \Rightarrow \left|\sum_{n=p}^{p} a_n\right| < \varepsilon$ or $|a_p| < \varepsilon$

$\therefore \forall \varepsilon > 0, \exists N \in \mathbb{R}$, s.t. if $p > N$, then $|a_p| < \varepsilon$,

or $|a_p - 0| < \varepsilon \Rightarrow \lim_{p\to\infty} a_p = 0$, or $\lim_{n\to\infty} a_n = 0$ $\boxtimes$

Ex 1: $\sum_{n=0}^{\infty} (-1)^n$ – d.n.c., because $\lim_{n\to\infty} (-1)^n$ – d.n.e.

Ex 2: $\sum_{n=0}^{\infty} \frac{n+1}{2n+3}$ – d.n.c., because $\lim_{n\to\infty} \frac{n+1}{2n+3} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2}$

$\neq 0$

Ex 3: $\sum_{n=0}^{\infty} \frac{1}{n}$ – harmonic series (diverges), but $\lim_{n\to\infty} \frac{1}{n} = 0$

Ex 4: $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ – geometric series with $|r| = \frac{3}{4} < 1 \Rightarrow$ converges.

$\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0 \Rightarrow$

Div. Test is inconclusive

Thm (Comparison test): Suppose $a_n \geq 0$ for all n.
Then:

(1) If $\sum_{n=0}^{\infty} a_n$ converges and $|b_n| \leq a_n$ for all $n \Rightarrow$ $\sum_{n=0}^{\infty} b_n$ converges.

(2) If $\sum_{n=0}^{\infty} a_n$ diverges, i.e., $\sum_{n=0}^{\infty} a_n = \infty$, and $b_n \geq a_n$ for all $n \Rightarrow \sum_{n=0}^{\infty} b_n = \infty$.

Proof: $\sum_{n=0}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=0}^{\infty} a_n$ satisfies the Cauchy criterion $\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{R}$ s.t. $|\sum_{n=l}^{m} a_n| < \varepsilon$ if $m \geq l > N \Leftrightarrow |\sum_{n=l}^{m} b_n| = |b_l + b_{l+1} + \dots + b_m|$
$\overset{T.I.}{\leq} |b_l| + |b_{l+1}| + \dots + |b_m| \underset{\forall n}{\overset{|b_n| \leq a_n}{\leq}} a_l + \dots + a_m = \sum_{n=l}^{m} a_n$
$\underset{\forall n}{\overset{a_n \geq 0}{=}} |\sum_{n=l}^{m} a_n| < \varepsilon$ if $m \geq l > N$. $\therefore \sum_{n=0}^{\infty} b_n$ satisfies the Cauchy Criterion $\Rightarrow \sum_{n=0}^{\infty} b_n$ converges.

(3) Suppose that $\sum_{n=0}^{\infty} a_n = \infty \Rightarrow \lim_{k \to \infty} \sum_{n=0}^{k} a_n = \infty$

However, because $b_n \geq a_n$ for all $n \Rightarrow \sum_{n=0}^{k} a_n \leq \sum_{n=0}^{k} b_n$

$\Rightarrow \lim_{k \to \infty} \sum_{n=0}^{\infty} b_n = \infty$ by the Comparison Thm for limits. ⊠

Def: $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges.

Def: $\sum_{n=0}^{\infty} a_n$ converges conditionally if $\sum_{n=0}^{\infty}$ converges but $\sum_{n=0}^{\infty} |a_n|$ diverges.

Thm: Suppose that $\sum_{n=0}^{\infty} a_n$ converges absolutely $\Rightarrow$ $\sum_{n=0}^{\infty} a_n$ converges.

Proof: We have that $\sum_{n=0}^{\infty} |a_n|$ converges. Clearly $|a_n| = |a_n|$ for all $n \underset{C.T.}{\Rightarrow} \sum_{n=0}^{\infty} a_n$ converges.

Thm (Root Test): Suppose that $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \alpha$, then

(1) If $\alpha < 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ converges absolutely

(2) If $\alpha > 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges or does not converge

(3) If $\alpha = 1 =$ test is inconclusive.

Proof: (1) $\alpha < 1 \Rightarrow$ there exists an $\varepsilon > 0$ small enough, so that $\alpha + \varepsilon < 1 \Rightarrow$ because $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \alpha$, there exists a $N \in \mathbb{R}$ s.t.

$\left|\sup_{n\geq k} |a_n|^{1/n} - \alpha\right| < \varepsilon$ if $k > N \iff -\varepsilon < \sup_{n\geq k} |a_n|^{1/n} - \alpha < \varepsilon$

if $k > N \iff \alpha - \varepsilon < \sup_{n \geq k} |a_n|^{1/n} < \alpha + \varepsilon$ if $k > N$

$\Rightarrow \sup_{n \geq N+1} |a_n|^{1/n} < \alpha + \varepsilon \iff \sup_{n > N} |a_n|^{1/n} < \alpha + \varepsilon$

$\iff |a_n|^{1/n} < \alpha + \varepsilon$ for <u>any</u> $n > N$. $\iff$

$|a_n| < (\underbrace{\alpha+\varepsilon}_{<1})^n$ for any $n > N$; $\alpha + \varepsilon < 1 \Rightarrow$

$\sum_{n=N+1}^{\infty} (\alpha+\varepsilon)^n$ – converges.

$\Downarrow$ C.T. $\Swarrow$

$\sum_{n=N+1}^{\infty} |a_n|$ converges.

$\Downarrow$

$$\sum_{n=0}^{\infty} |a_n| = \underbrace{\sum_{n=0}^{N} |a_n|}_{\text{finite sum}} + \underbrace{\sum_{n=N+1}^{\infty} |a_n|}_{\text{convergent}}$$

– the entire series converges.

(2) Suppose that $\alpha > 1$, i.e., $\overline{\lim_{n\to\infty}} |a_n|^{1/n} > 1$. Then there exists a subsequence $\{a_{n_k}\}$ s.t. $\lim_{k\to\infty} |a_{n_k}|^{1/n_k} = \alpha > 1 \Rightarrow$ starting from some $K > 0$, $|a_{n_k}|^{1/n_k} > 1$ $\Rightarrow |a_{n_k}| > 1^{n_k} = 1$ for all $k > K$. Now, if we assume that

$\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{k\to\infty} |a_{n_k}| = 0$; contradicts

$|a_{n_k}| > 1$ for all $k > K$.

$\therefore \lim_{n\to\infty} |a_n| \neq 0 \Rightarrow \lim_{n\to\infty} a_n \neq 0 \underset{\text{for DIV}}{\overset{\text{Test}}{\Rightarrow}} \sum_{n=0}^{\infty} a_n$

does not converge.

(3) $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1$:

<u>Ex A:</u> $a_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}}$

$= \lim_{n\to\infty} 1 \Big/ \lim_{n\to\infty} n^{1/n} = 1/1 = 1$

$\Rightarrow \overline{\lim_{n\to\infty}} \left(\frac{1}{n}\right)^{1/n} = 1 \Rightarrow \alpha = 1$

In this case, $\sum_{n=1}^{\infty} \frac{1}{n}$ – diverges because this is a harmonic series.

<u>Ex B:</u> $a_n = \frac{1}{n^2} \Rightarrow \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{(n^2)^{1/n}}$

$= \lim_{n\to\infty} \frac{1}{n^{2/n}} = \lim_{n\to\infty} \frac{1}{(n^{1/n})^2} = 1 \Rightarrow \overline{\lim_{n\to\infty}} \left(\frac{1}{n^2}\right)^{1/n} = 1$

$\Rightarrow \alpha = 1$; However, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. ⊠

Thm (Ratio Test): The series $\sum a_n$

(1) Converges absolutely if $\overline{\lim\limits_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1$

(2) Diverges if $\underline{\lim\limits_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| > 1$

(3) The test is inconclusive if $\overline{\lim\limits_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| \geq 1$ and $\underline{\lim\limits_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| \leq 1$.

Proof: Recall that we proved:

$$\underline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| \leq \underline{\lim_{n\to\infty}} |a_n|^{1/n} \leq \overline{\lim_{n\to\infty}} |a_n|^{1/n} \leq \overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right|$$

$\Rightarrow$

- If $\overline{\lim\limits_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \overline{\lim\limits_{n\to\infty}} |a_n|^{1/n} < 1 \Rightarrow \sum\limits_{n=0}^{\infty} a_n$ converges absolutely by the Root test.
- If $\underline{\lim\limits_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \overline{\lim\limits_{n\to\infty}} |a_n|^{1/n} > 1 \Rightarrow \sum\limits_{n=0}^{\infty} a_n$ diverges by the Root Test.
- To prove (3), again, use the examples $a_n = \frac{1}{n}$ and $a_n = \frac{1}{n^2}$ $\boxtimes$

Since we used the Root Test to prove the Ratio test, the Root Test is stronger, i.e., there are examples when the Root Test works, but the Ratio Test is inconclusive. If the Root test is inconclusive $\Rightarrow$ Ratio test is inconclusive: indeed, if

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1 \implies$$

$$\underline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| \leq \underline{\lim_{n\to\infty}} |a_n|^{1/n} \leq 1 \leq \overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| \implies$$

$$\underline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right|.$$

There is, however, the case when both tests are equivalent:

<u>Corollary:</u> If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \implies \lim\limits_{n\to\infty} |a_n|^{1/n} = L$ and both the Ratio and the Root tests are equivalent.

<u>Proof:</u> If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \implies \overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| = \underline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| = L.$

Because

$$L = \underline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| \leq \underline{\lim_{n\to\infty}} |a_n|^{1/n} \leq \overline{\lim_{n\to\infty}} |a_n|^{1/n} \leq \overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| = L$$

$$\implies \underline{\lim_{n\to\infty}} |a_n|^{1/n} = \overline{\lim_{n\to\infty}} |a_n|^{1/n} = L \implies \lim_{n\to\infty} |a_n|^{1/n} = L \quad \boxtimes$$

<u>Ex.1:</u> Does the series $\sum\limits_{n=0}^{\infty} 2^{(-1)^n - n}$ converge?

(1) Use the Ratio Test: $\overline{\lim\limits_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| = \overline{\lim\limits_{n\to\infty}} \frac{2^{(-1)^{n+1} - (n+1)}}{2^{(-1)^n - n}}$

$$= \overline{\lim_{n\to\infty}} 2^{(-1)^{n+1} - (-1)^n - (\cancel{n}+1) + \cancel{n}} = \overline{\lim_{n\to\infty}} 2^{2(-1)^{n+1} - 1}$$

The sequence $2^{2(-1)^{n+1}-1}$ looks like:

$$2^{-3}, 2, 2^{-3}, 2, \ldots \Rightarrow \overline{\lim_{n\to\infty}} 2^{2(-1)^{n+1}-1} = 2 \geq 1$$

At the same time, $\underline{\lim_{n\to\infty}} 2^{2(-1)^{n+1}-1} = 2^{-3} \leq 1$

- Ratio Test is inconclusive

(2) Use the Root Test:

$$\overline{\lim_{n\to\infty}} \left|2^{(-1)^n-n}\right|^{1/n} = \overline{\lim_{n\to\infty}} 2^{\frac{(-1)^n-n}{n}} = \overline{\lim_{n\to\infty}} 2^{\frac{(-1)^n}{n}-1} = \frac{1}{2} < 1$$

$\Rightarrow$ the series converges absolutely by the Root test.

<u>Ex.</u> Does the series $\sum_{n=0}^{\infty} \frac{n+1}{3n-2}$ converge?

$\lim_{n\to\infty} \frac{n+1}{3n-2} = \frac{1}{3} \neq 0$ - the series diverges by Divergence test.

<u>Ex.</u> Does the series $\sum_{n=0}^{\infty} \frac{n+1}{3n^2-2}$ converge?

$\lim_{n\to\infty} \frac{n+1}{3n^2-2} = \lim_{n\to\infty} \frac{\frac{n}{n^2}+\frac{1}{n^2}}{3-\frac{2}{n^2}} = 0 \Rightarrow$ Divergence test is inconclusive

Try the Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)+1}{3(n+1)^2-2}}{\frac{n+1}{3n^2-2}} \right| = \lim_{n\to\infty} \frac{n+2}{3n^2+6n+1} \frac{3n^2-2}{n+1} = \lim_{n\to\infty} \frac{n\cdot 3n^2}{3n^2\cdot n}$$

$= 1$ – inconclusive $\Rightarrow$ By the corollary above,

$\lim_{n\to\infty} \left|\frac{n+1}{3n-2}\right|^{1/n} = 1 \Rightarrow$ Root test is inconclusive.

Try comparison test: Motivation: when n is large, $\frac{n+1}{3n^2-2} \approx \frac{1}{3n}$

and $\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges. Now, $\frac{n+1}{3n^2-2} > \frac{n}{3n^2-2} > \frac{n}{3n^2} = \frac{1}{3n}$, if $n \geq 1$

$\Rightarrow \sum_{n=0}^{\infty} \frac{n+1}{3n^2-2}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{3n}$.

<u>In general:</u> If a_n is an algebraic expression in terms of n (involving constant powers, roots, fractions) $\Rightarrow$ Both Ratio and Root tests are inconclusive!!!

<u>Ex:</u> Does the series $\sum_{n=5}^{\infty} \left(\frac{1}{5}\right)^n$ converge? Yes, this is a geometric series with $r = \frac{1}{5}$.

<u>Ex:</u> Does the seris $\sum_{n=5}^{\infty} n\left(\frac{1}{5}\right)^n$? Use the Ratio test:

$\lim_{n\to\infty} \frac{(n+1)\left(\frac{1}{5}\right)^{n+1}}{n\left(\frac{1}{5}\right)^n} = \frac{1}{5}\lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{5} < 1$ – the series converges absolutely.

<u>Ex.</u> $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ where $n! = 1\cdot 2 \cdots (n-1)\cdot n$ – does this

series converge? Use Ratio test:

$$\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \lim_{n\to\infty} \frac{\cancel{3^n}\cdot 3\cdot \cancel{n}\cdot\cancel{(n-1)}\cdot\ldots\cdot\cancel{2}\cdot\cancel{1}}{(n+1)\cdot\cancel{n}\cancel{(n-1)}\cdot\ldots\cdot\cancel{2}\cdot\cancel{1}\cdot\cancel{3^n}} = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1$$

$\Rightarrow$ The series converges by the Ratio test.