I will illustrate the next test by using the following example. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ – does it converge? We can try to use the tests from the previous section:

(1) $\lim_{n\to\infty} \frac{1}{n^2} = 0$ – Divergence test is inconclusive.

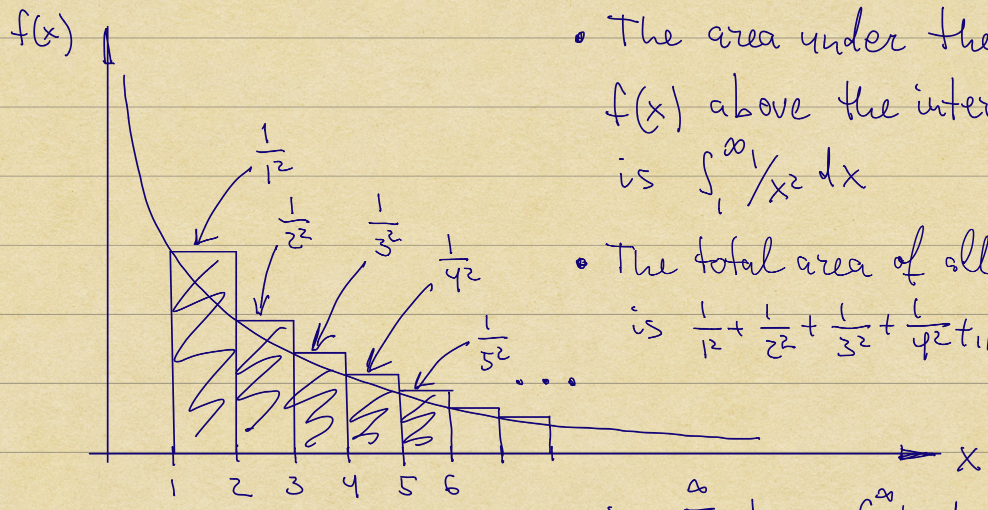
(2) $\frac{1}{n^2}$ is a rational function of n – both Root and Ratio test will be inconclusive.

(3) Comparison test: the only reasonable comparison is with $\frac{1}{n}$, however $\frac{1}{n} \geq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow$ comparison is inconclusive.

We need to come up with a different test...

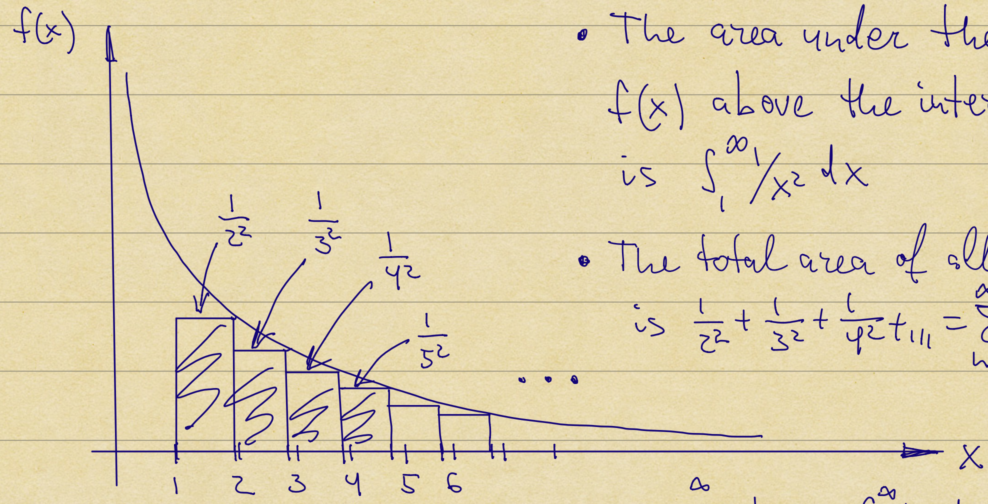
Our series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$; if we let $f(x) = \frac{1}{x^2} \Rightarrow a_n = f(n) = \frac{1}{n^2}$

Consider $f(x)$ on the interval $(0, \infty)$:



- The area under the graph of $f(x)$ above the interval $[1, \infty)$ is $\int_1^{\infty} 1/x^2 \, dx$
- The total area of all rectangles is $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} > \int_1^{\infty} \frac{1}{x^2} \, dx$$



- The area under the graph of $f(x)$ above the interval $[1,\infty)$ is $\int_1^\infty 1/x^2\,dx$
- The total area of all rectangles is $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\sum_{n=2}^{\infty}\frac{1}{n^2}$

$$\Rightarrow \sum_{n=2}^{\infty}\frac{1}{n^2} < \int_1^\infty \frac{1}{x^2}\,dx$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} < 1+\int_1^\infty \frac{1}{x^2}\,dx$$

We conclude that

$$\int_1^\infty \frac{1}{x^2}\,dx < \sum_{n=1}^{\infty}\frac{1}{n^2} < 1+\int_1^\infty \frac{1}{x^2}\,dx \qquad (*)$$

and, therefore, either both the integral and the series diverge at the same time or converge at the same time.

Now, $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t x^{-2}\,dx = \lim_{t\to\infty}\left(-x^{-1}\Big|_1^t\right) = \lim_{t\to\infty}\left(1-\frac{1}{t}\right)=1$

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges; further, from $(*)$ $1<\sum_{n=1}^{\infty}\frac{1}{n^2}<2$.

We can now formulate the Integral test:

Thm (Integral Test): Suppose that $f(x)$ is

(1) Decreasing on $[1,\infty)$

(2) $\lim_{x\to\infty} f(x) = 0$

Then, the series $\sum_{n=1}^{\infty} f(n)$ converges if $\int_1^{\infty} f(x)dx$ converges.

Ex: For what values of p does the series $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^p}}_{\text{called a } p\text{-series.}}$ converges?

(1) If $p \leq 0 \Rightarrow \frac{1}{n^p} = n^{-p} \not\to 0$ as $n \to \infty$ ← positive or 0

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $p \leq 0$

(2) Suppose that $p > 0 \Rightarrow$ the function $f(x) = \frac{1}{x^p}$ is

(a) decreasing on $[1,\infty)$, because
$f'(x) = -p x^{-p-1} < 0$

(b) $\lim_{x\to\infty} \frac{1}{x^p} = 0$

$\Rightarrow$ Integral test applies; Now compute

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t x^{-p} dx \overset{p\neq 1}{=} \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t$$

$$= \frac{1}{1-p} \lim_{t\to\infty} \left(t^{1-p} - 1\right) = \begin{cases} \infty, & p < 1 \\ \frac{1}{p-1}, & p > 1 \end{cases}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$ and converges if $p > 1$

($p = 1$ is the harmonic series)

Ex: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ - converges?

Check: Test for divergence, Root and Ratio tests are inconclusive.

Try the Integral test: $f(x) = \frac{1}{x(\ln x)^2} = \left[x(\ln x)^2\right]^{-1}$

(1) $f'(x) = -\left(x(\ln x)^2\right)^{-2}\left((\ln x)^2 + 2\cancel{x}\ln x \cdot \frac{1}{\cancel{x}}\right)$

$= -\left(x(\ln x)^2\right)^{-2}\left((\ln x)^2 + 2\ln x\right) = -\frac{\ln x}{x^2(\ln x)^4}(\ln x + 2)$

< 0 if $x \geq 2 \Rightarrow f$ is $\downarrow$ on $[2, \infty)$

(2) $0 \leq \lim_{x\to\infty} \frac{1}{x(\ln x)^2} \leq \lim_{x\to\infty} \frac{1}{x} = 0 \Rightarrow \lim_{x\to\infty} \frac{1}{x(\ln x)^2} = 0$

$\Rightarrow$ Integral Test applies.

(3) $\int_2^{\infty} \frac{dx}{x(\ln x)^2} \underset{du = \frac{1}{x}dx}{\overset{u = \ln x}{=}} \int_{\ln 2}^{\infty} \frac{du}{u^2} = \left(-\frac{1}{u}\right)\Big|_{\ln 2}^{\infty} = \frac{1}{\ln 2}$

$\Rightarrow$ the integral and thus the series converges.