A series of the form $\sum_{n=0}^{\infty}(-1)^n a_n$, where $a_n \geq 0$, $\forall n \in \mathbb{N}$ is called an alternating series.

Alternating Series Test: Suppose that $a_n > 0$ for all $n \in \mathbb{N}$ and (1) the sequence $\{a_n\}$ is decreasing; (2) $\lim_{n\to\infty} a_n = 0$. Then, the alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

<u>Proof</u>: We need to show that the sequence of partial sums $s_k = \sum_{n=0}^{k}(-1)^n a_n$ converges as $k \to \infty$.

(1) Consider the subsequence of partial sums of even number of terms: $s_1, s_3, s_5, \ldots$ that we can denote as $\{s_{2l+1}\}_{l=0}^{\infty}$. Then $s_1 = a_0 - a_1$, $s_3 = a_0 - a_1 + a_2 - a_3$, etc.

The difference between any two consequtive term of this sequence is

$$s_{2(l+1)+1} - s_{2l+1} = \sum_{n=0}^{2(l+1)+1}(-1)^n a_n - \sum_{n=0}^{2l+1}(-1)^n a_n = \sum_{n=2l+2}^{2l+3}(-1)^n a_n$$

$$= \underbrace{(-1)^{2l+2}}_{\text{even}} a_{2l+2} + \underbrace{(-1)^{2l+3}}_{\text{odd}} a_{2l+3} = a_{2l+2} - a_{2l+3} \geq 0,$$

because we assumed that $\{a_n\}$ is decreasing. $\therefore \{s_{2l+1}\}$ is $\uparrow$

Further, $s_{2l+1} = a_0 \underbrace{-a_1 + a_2}_{\leq 0} \underbrace{-a_3 + a_4}_{\leq 0} + \ldots \underbrace{-a_{2l-1} + a_{2l}}_{\leq 0} \underbrace{-a_{2l+1}}_{\leq 0} \leq a_0$

for all $l \in \mathbb{N}$ because $\{a_n\}$ is $\downarrow$. $\therefore$ $\{s_{2l+1}\}$ is bounded from above by a_0 $\Rightarrow$ $\{s_{2l+1}\}$ is $\uparrow$ and bounded from above $\Rightarrow$ $\{s_{2l+1}\}$ converges as $l \to \infty$

(2) Now consider the sequence of partial sums that contain odd # of elements: $s_0, s_2, s_4, \ldots$ This can be denoted by $\{s_{2l}\}_{l=0}^{\infty}$. Then $s_0 = a_0$, $s_2 = a_0 - a_1 + a_2$, $s_3 = a_0 - a_1 + a_2 - a_3 + a_4$, ets.

The difference between any two consequtive term of this sequence is

$$s_{2(l+1)} - s_{2l} = \sum_{n=0}^{2(l+1)} (-1)^n a_n - \sum_{n=0}^{2l} (-1)^n a_n = \sum_{n=2l+1}^{2l+2} (-1)^n a_n$$

$$= (-1)^{\underset{\text{odd}}{2l+1}} a_{2l+1} + (-1)^{\underset{\text{even}}{2l+2}} a_{2l+2} = -a_{2l+1} + a_{2l+2} \leq 0$$

because we assumed that $\{a_n\}$ is decreasing. $\therefore$ $\{s_{2l}\}$ is $\downarrow$

Further, $s_{2l} = \underbrace{a_0 - a_1}_{\geq 0} + \underbrace{a_2 - a_3}_{\geq 0} + \ldots + \underbrace{a_{2l} - a_{2l+1}}_{\geq 0} + \underbrace{a_{2l+2}}_{\geq 0} \geq 0$

for all $l \in \mathbb{N}$ because $\{a_n\}$ is $\downarrow$. $\therefore$ $\{s_{2l}\}$ is bounded from below by 0 $\Rightarrow$ $\{s_{2l}\}$ is $\downarrow$ and bounded from below $\Rightarrow$ $\{s_{2l}\}$ converges as $l \to \infty$.

(3) Now, consider $\lim\limits_{l\to\infty} s_{2l+1} - \lim\limits_{l\to\infty} s_{2l} = \lim\limits_{l\to\infty} (s_{2l+1} - s_{2l})$

$= \lim\limits_{l\to\infty} \left(\sum\limits_{n=0}^{2l+1} (-1)^n a_n - \sum\limits_{n=0}^{2l} (-1)^n a_n \right) = \lim\limits_{l\to\infty} (-1)^{2l+1} a_{2l+1}$

$= -\lim\limits_{l\to\infty} a_{2l+1} = 0$, because we assumed that $\lim\limits_{n\to\infty} a_n = 0$.

$\Rightarrow$ both even and odd partial sums converge to the same limit $\Rightarrow \{s_n\}$ converges $\Rightarrow \sum\limits_{n=0}^{\infty} (-1)^n a_n$ converges. $\boxtimes$

<u>Ex.</u> Does the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge? If yes, does it converge absolutely or conditionally?

The series of absolute values is $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ – harmonic series, diverges. Hence if the original series converges, it does so conditionally.

$\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series; further $a_n = \frac{1}{n} \Rightarrow a_n \downarrow$ and $\lim\limits_{n\to\infty} a_n = 0 \Rightarrow \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally by the Alternating Series test.

<u>Ex:</u> Does the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge? If yes, does it converge absolutely or conditionally?

The series of absolute values is $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ –

this is a p-series with $p = 2 > 1 \Rightarrow$ the series of absolute values converges $\Rightarrow$ the alternating series converges absolutely (and thus converges - no need to use the Alternating series test).