The purpose of this class is to provide rigorous underpinnings to elementary calculus. The main idea is to identify the minimal set of assumptions that will enable us to establish various facts about functions and their behavior and to define notions like a limit, continuity, derivative, etc. The initial assumptions are known as axioms, and logically provable facts that follow from them are theorems.

Since we will be dealing with numbers throughout this course, we beging with their axiomatic definition.

I. Natural (counting) #'s:

Since these are used for counting, we need a concept of a singe object ore "one". Also, if we can count objects to, say, 3 we should be able to add one more object to get 4. In other words, every counting number should have the next counting # or a <u>successor</u> for us to be able to keep counting. Next, if two numbers have the same succesor, they actually should be the same number: for counting to be meaningful it makes no sense to, say, 5 appear after 4 and 51. If we are counting something, then the least number of objects you might have is 1, that is 1 is not a successor of any counting #.

Finally, our counting process cannot split into several branches at some point: each # can only have one successor. Then, if we start from 1 and keep counting, we should eventually go through all counting #'s (it would take an infinite amount of time, of course :))

These observations allow us to define the set of natural #'s $\mathbb{N}$ via the following set of axioms that we assume to be true:

(1) 1 belongs to $\mathbb{N}$;

(2) If $n$ belongs to $\mathbb{N}$, then $n+1$ belongs to $\mathbb{N}$;
(3) 1 is not a successor of any $n$ in $\mathbb{N}$;
(4) If $n$ and $m$ have the same successor, then $n=m$;
(5) Suppose that $S$ is a subset of $\mathbb{N}$ such that (a) 1 belongs to $S$

and (b) If $n$ is in $S$, then $n+1$ is in $S$. Then $S = \mathbb{N}$.

We are now allowed to think that the set of natural #'s $\mathbb{N}$ satisfies the properties (1) - (5) and we can logically deduce other properties that follow from these ones.

Before illustrating how we can use (1) - (5) in practice let's agree on the following notation:

(*) $a \in S$ means that $a$ is an element of $S$.

(*) $a \notin S$ means that $a$ is not an element of $S$.

(*) $S \subset T$ means that the set $S$ is a subset of the set $T$, i.e., if $a \in S \Rightarrow a \in T$. Here "$\Rightarrow$" means "then".

(*) $\forall a \in S$ means "for any $a$ that is an element of $S$".

(*) $\exists a \in S$ means "there exists an element $a$ of $S$".

(*) $:$ means "such that".

Using this notation, we can rewrite the axiom (5) above as:

(5) Suppose that $S \subset \mathbb{N}$ : (a) $1 \in S$ and (b) $n \in S \Rightarrow n+1 \in S$.
Then $S = \mathbb{N}$

We will also use similar notation to describe infinite sets, e.g., the set $A$ of all even counting numbers can be denoted in the following way:

$$A = \{n \in \mathbb{N} : \exists k \in \mathbb{N} \text{ s.t. } n = 2k\}$$

that is

"$A$ is the set of all natural numbers $n$ s.t. there exists another natural number $k$ satisfying $n = 2k$"

which indeed means that $n$ is even.

Now, let's return to applications of the axioms of natural numbers:

Suppose we are given the list of infinitely many statements $P_1, P_2, P_3, \dots$ that may or may not be true. For example, this list may look like:

$\overset{P_1}{\text{"}2>1\text{"}}$, $\overset{P_2}{\text{"}3>2\text{"}}$, $\overset{P_3}{\text{"}4>3\text{"}}$, ..., $\overset{P_n}{\text{"}n+1>n\text{"}}$, ...

– clearly all of these statements are true.

Second example:

$\overset{P_1}{\text{"1 is divisible by 2"}}$, $\overset{P_2}{\text{"2 is divisible by 2"}}$, $\overset{P_3}{\text{"3 is divisible by 2"}}$,

$\ldots$, "n is divisible by 2" ($P_n$), $\ldots$

– every other statement is true.

Third example:

"$1+2=\frac{2\cdot 3}{2}$" ($P_1$), "$1+2+3=\frac{3\cdot 4}{2}$" ($P_2$), "$1+2+3+4=\frac{4\cdot 5}{2}$" ($P_3$), $\ldots$,

"$1+2+\ldots+(n-1)+n=\frac{n(n+1)}{2}$" ($P_n$), $\ldots$

while the first three statements can be easily verified, what about the remaining infinitely many statements?

To decide whether all of the infinitely many statements are true, we will use the method of mathematical induction that I will now describe.

Returning to our list of infinitely many statements $P_1, P_2, P_3, \ldots$, consider all statements that are true and collect their indices in a set that we will denote by $S$ (e.g., if only $P_1$, $P_7$, and $P_{531}$ are true and the rest of the statements are false, then $S=\{1,7,531\}$). Observe that $S$ is a subset of the set of all natural #'s, i.e., $S\subset\mathbb{N}$.

Now, suppose that we know that (a) $P_1$ is true and (b) if $P_n$ is true, then $P_{n+1}$ is true. This means that

$$(a)\ 1 \in S \text{ and } (b) \text{ if } n \in S \Rightarrow n+1 \in S$$

implying that $S$ satisfies the conditions of the axiom (5) $\Rightarrow S = \mathbb{N}$, therefore the statements $P_n$ are true for all $n \in \mathbb{N}$.

We can summarize mathematical induction as follows: suppose that

(1) $P_1$ is true (basis of induction)

(2) If $P_n$ is true, then $P_{n+1}$ is true (induction step)

then $P_n$ is true for all $n \in \mathbb{N}$.

Example 1: Prove that $1+2+\dots+(n-1)+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

(1) Basis of induction: $P_1$: $1 = \frac{1(1+1)}{2} = 1$ - true.

(2) Induction step: Suppose that $P_n$ is true, that is $1+2+\dots+(n-1)+n = \frac{n(n+1)}{2}$. We want to use this information to prove that $P_{n+1}$

is true. The statement $P_{n+1}$ is

$$1+2+\dots+(n-1)+n+(n+1)=\frac{(n+1)((n+1)+1)}{2}.$$

Let's check if this is true:

$$\underbrace{1+2+\dots+(n-1)+n}_{\text{use } P_n}+(n+1) \overset{P_n \text{ is}}{\underset{\text{true}}{=}} \frac{n(n+1)}{2}+(n+1)$$

$$=(n+1)\left(\frac{n}{2}+1\right)=(n+1)\frac{n+2}{2}$$

$$=\frac{(n+1)((n+1)+1)}{2} \Rightarrow P_{n+1} \text{ is true}$$

and the induction step holds. We conclude that $P_n$ is true for all $n \in \mathbb{N}$ by mathematical induction.

Example 2: $3^n-2n-1$ is divisible by 4 for all $n \in \mathbb{N}$.

Use mathematical induction;

(1) Basis of induction: $n=1 \Rightarrow 3^1-2\cdot1-1=0$ and 0 is divisible by 4 - ok.

(2) Induction step: Assume $3^n-2n-1$ is divisible by 4 $\Rightarrow \exists k \in \mathbb{N}$ s.t. $3^n-2n-1=4k$. Need to show that

$3^{n+1} - 2(n+1) - 1$ is divisible by 4.

We have $3^{n+1} - 2(n+1) - 1 = 3 \cdot 3^n - 2n - 2 - 1$

$= 3(4k + 2n + 1) - 2n - 3 = 12k + 6n + \cancel{3} - 2n - \cancel{3}$

$= 12k + 4n = 4\underbrace{(3k + n)}_{\in \mathbb{N}}$ - divisible by 4 - OK

$\Rightarrow 3^n - 2n - 1$ is divisible by 4 for all $n \in \mathbb{N}$. $\blacksquare$

Example 3: Show that $5^n - 3^n$ is divisible by 2 for all $n \in \mathbb{N}$.

(1) Basis of induction: $5^1 - 3^1 = 2$ - divisible by 2 - OK.

(2) Induction step: Assume that $5^n - 3^n$ is divisible by 2, that is, there exists a $k \in \mathbb{N}$ s.t. $5^n - 3^n = 2k$. Need to show that $5^{n+1} - 3^{n+1}$ is divisible by 2.

$5^n = 3^n + 2k \Rightarrow$ $5^{n+1} - 3^{n+1} = 5 \cdot 5^n - 3^{n+1}$

$= 5(3^n + 2k) - 3^{n+1} = 5 \cdot 3^n + 10k$

$- 3 \cdot 3^n = 3^n(5 - 3) + 10k$

$= 2 \cdot 3^n + 10k = 2\underbrace{(3^n + 5k)}_{\in \mathbb{N}}$

- divisible by 2 - OK

We conclude using mathematical induction that $5^n - 3^n$ is divisible by 2 for all $n \in \mathbb{N}$.

Example 4: $|\sin nx| \leq n|\sin x|$ for all $n \in \mathbb{N}$.

Use induction:

(1) Basis of induction: $|\sin x| \leq |\sin x|$ — true

$\Rightarrow$ ok

(2) Induction step: Assume that

$$|\sin nx| < n|\sin x|$$

Want to show that $|\sin(n+1)x| < (n+1)|\sin x|$ ?

$|\sin(n+1)x| = |\sin(nx+x)| = |\sin nx \cos x$

$+ \cos nx \sin x| \leq |\sin nx \cos x| + |\cos nx \sin x|$

$= |\sin nx| \underbrace{|\cos x|}_{\leq 1} + \underbrace{|\cos nx|}_{\leq 1} |\sin x| \leq |\sin nx|$

$+ |\sin x| < n|\sin x| + |\sin x| = (n+1)|\sin x|$ — ok

$\Rightarrow$ $|\sin nx| < n|\sin x|$ for all $n \in \mathbb{N}$ by mathematical induction. $\boxtimes$