Fact: a rational # can be represented by a terminating or repeating decimal; the opposite is also true:

Ex. 1: $x = 3.213 \Rightarrow x = \frac{3213}{1000}$ - rational

Ex. 2: $x = 3.\overline{213} = 3.213213213\ldots$ - to show that $x \in \mathbb{Q}$:

$$1000x = 3213.213213\ldots = 3213.\overline{213} \Rightarrow$$

$$999x = 1000x - x = 3213.\overline{213} - 3.\overline{213} = 3210 \Rightarrow x = \frac{3210}{999}$$

- this is a rational #

Lets "complete" the set of rational #'s in the decimal representation by including the nonrepeating, non-terminating decimals, that we will call <u>irrational</u> numbers. We denote the resulting, larger set of #'s by $\mathbb{R}$ and call there real #'s. Thus $\mathbb{Q} \subset \mathbb{R}$.

(1) $x^2 = 2 \rightarrow 1.41\ldots$ - not rational; on the other hand $1.4, 1.41, 1.42$ are $\Rightarrow$ we can find rational #'s arbitrarily close to x.

(2) $x = 1.234567\ldots$ - not a rational #; if it was then $x = \frac{p}{q} \Rightarrow x$ is a terminating or repeating decimal and x is neither. However:

$x_1 = 1,\ x_2 = 1.2,\ x_3 = 1.23\ \ x_4 = 1.234$, etc. are rational $\Rightarrow x - x_k > 0$ and is 0 up to the k-1 decimal $\Rightarrow$ can find a rational # that is less than x and arbitrarily close to x.

Also:

$y_1 = 2$, $y_2 = 1.3$, $y_3 = 1.24$, $y_4 = 1.235$,

$y_5 = 1.2346$, $y_6 = 1.23457$, $y_7 = 1.234568$, $y_8 = 1.2345679$

$y_9 = 1.2345678911$, etc.

$\Rightarrow$ $x - y_k < 0$ and is 0 up to the k-th decimal $\Rightarrow$ can find a rational # that is greater than x and arbitrarily close to x.

$\therefore$ There are "holes" in the set of rational #'s that can be filled with non-terminating, non-repeating decimals representing irrational numbers.

We can define addition, subtraction, multiplication, division of decimals in the usual way to observe that $\mathbb{R}$, just like $\mathbb{Q}$ is closed under these operations.

Hence the distinction between the properties of $\mathbb{R}$ and $\mathbb{Q}$ is <u>not</u> due to there properties arising from arithmetic operations. One distinction that we already observed is that $\mathbb{Q}$ does not contain all algebraic #'s; neither does $\mathbb{R}$ (just think of $x^2+1=0$), but $\mathbb{R}$ includes many more of these #'s (e.g., $\sqrt{2} \notin \mathbb{Q}$ but $\sqrt{2} \in \mathbb{R}$). In the next lecture we will note another importand distinction between $\mathbb{Q}$ and $\mathbb{R}$ related to the fact that $\mathbb{Q}$ contains holes.

For the rest of this lecture, we concentrate on common properties of $\mathbb{R}$ and $\mathbb{Q}$ that stem from arithmetic operations.

Suppose that V may denote either $\mathbb{R}$ or $\mathbb{Q}$. On V we define addition and scalar multiplication with usual rules; the minimal set of rules is specified in the following axioms. All other properties will stem from these axioms.

<u>Axioms of a field</u>: Suppose that addition and multiplication are defined on V. Then V is a <u>field</u> if the following hold:

A1. $a+(b+c)=(a+b)+c$, $\forall a,b,c \in V$
 – associative law

A2. $a+b=b+a$, $\forall a,b \in V$
 – commutative law

A3. $a+0=a$, $\forall a \in V$

A4. $\forall a \in V$, $\exists -a \in V$ s.t. $a+(-a)=0$

M1. $a(bc)=(ab)c$, $\forall a,b,c \in V$
 – associative law

M2. $ab=ba$ for any $a,b \in V$
 – commutative law

M3. $a \cdot 1 = a$ for all $a \in V$

M4. $\forall a \in V$ s.t. $a \neq 0$, $\exists\, a^{-1} \in V$ s.t. $a \cdot a^{-1} = 1$

DL. $a(b+c) = ab + ac$, $\forall a, b, c \in V$
- distributive law.

The following Theorem holds:

<u>Thm</u>: Suppose $a, b, c, \ldots, \in V$ and V is a field.
Then the following are true:

(i) $a + c = b + c \Rightarrow a = b$

(ii) $a \cdot 0 = 0$

(iii) $(-a)b = -ab$

(iv) $(-a)(-b) = ab$

(v) $ac = bc$ and $c \neq 0 \Rightarrow a = b$

(vi) $ab = 0 \Rightarrow a = 0$ or $b = 0$

<u>Proof</u>: (i) Suppose that $a + c = b + c$.
Then, by A4, there exists $-c \in V$ s.t. $c + (-c) = 0$.

$$\Rightarrow (a+c) + (-c) = (b+c) + (-c)$$

$$\overset{A1 * 2}{\Rightarrow} a + (c + (-c)) = b + (c + (-c))$$

$$\overset{A4 * 2}{\Rightarrow} a + 0 = b + 0 \overset{A3 * 2}{\Rightarrow} a = b \quad \boxtimes$$

(ii) $a\cdot 0 \overset{A3}{=} a\cdot(0+0) \overset{DL}{=} a\cdot 0 + a\cdot 0$

Also, $a\cdot 0 + 0 \overset{A3}{=} a\cdot 0 \Rightarrow a\cdot 0 + 0 = a\cdot 0 + a\cdot 0$

$\overset{A2}{\Rightarrow} 0 + a\cdot 0 = a\cdot 0 + a\cdot 0 \overset{(i)}{\Rightarrow} 0 = a\cdot 0$ $\boxtimes$

(iii) $0 \underset{M2}{\overset{(ii)}{=}} 0\cdot b \overset{A4}{=} (a+(-a))b \overset{DL}{=} ab+(-a)b$

Also, $ab + (-ab) \overset{A4}{=} 0 \Rightarrow$

$ab+(-ab) = ab+(-a)b \overset{A2*2}{\Rightarrow}$

$-ab+ab = (-a)b+ab \overset{(i)}{\Rightarrow} -ab = (-a)b$ $\boxtimes$

(v) Suppose that $ac=bc$ and $c\neq 0$

$\overset{M4}{\Rightarrow} \exists\, c^{-1}$ s.t. $c^{-1}c=1 \Rightarrow$

$c^{-1}(ac)=c^{-1}(bc) \overset{M2*2}{\Rightarrow}$

$c^{-1}(ca)=c^{-1}(cb) \overset{M1*2}{\Rightarrow}$

$(c^{-1}c)a=(c^{-1}c)b \overset{M4}{\Rightarrow}$

$1\cdot a = 1\cdot b \overset{M3*2}{\Rightarrow} a=b$ $\boxtimes$

(vi) Assume $ab=0$. (1) Let $b\neq 0 \overset{(ii)}{\Rightarrow} b\cdot 0=0$

$\overset{M2}{\Rightarrow} ab=0b \overset{(v)}{\Rightarrow} a=0$

(2) Let $a\neq 0$ – the same

arguments as in (1) imply that $b=0$

(3) The only remaining choice is $a=b=0$

$\Rightarrow$ always at least one of a,b is zero $\boxtimes$

<u>Ordering:</u> In $\mathbb{Q}$ or $\mathbb{R}$ any two #'s can be compared: either $x_1 \le x_2$ or $x_2 \le x_1$.

A field V has an ordered structure (V is an ordered field) if

O1. $\forall a,b \in V$ either $a \le b$ or $b \le a$

O2: If $a \le b$ and $b \le a \Rightarrow a=b$

O3: $a \le b$ and $b \le c \Rightarrow a \le c$

– transitive law

O4: If $a \le b \Rightarrow a+c \le b+c$, $\forall c \in V$

O5: If $a \le b$ and $0 \le c \Rightarrow ac \le bc$

– axioms of ordering.

Note: We say: $b \ge a$ if $a \le b$

$a < b$ if $a \le b$ and $a \ne b$

$a > b$ if $b \le a$ and $a \ne b$

Theorem: Suppose that V is an ordered field and $a, b, c, \ldots \in V$. Then

(1) If $a \leq b$, then $-b \leq -a$;

(2) If $a \leq b$, and $c \leq 0$ then $bc \leq ac$;

(3) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$;

(4) $0 \leq a^2$ for $\forall a \in V$;

(5) $0 < 1$;

(6) If $0 < a$, then $0 < a^{-1}$;

(7) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$!

<u>Proof:</u> (1) Suppose that $a \leq b$. By A4 there exists $-a \in V$ s.t. $a + (-a) = 0$.

$$a \leq b \overset{O4}{\Longrightarrow} a + (-a) \leq b + (-a) \overset{A4}{\Longrightarrow}$$

$$0 \leq b + (-a) \underset{O4}{\overset{A4}{\Longrightarrow}} 0 + (-b) \leq (b + (-a)) + (-b)$$

$$\overset{A2*2}{\Longrightarrow} (-b) + 0 \leq ((-a) + b) + (-b) \overset{A3}{\Longrightarrow}$$

$$-b \leq ((-a) + b) + (-b) \overset{A1}{\Longrightarrow}$$

$$-b \leq -a + (b + (-b)) \overset{A4}{\Longrightarrow}$$

$$-b \leq -a + 0 \overset{A3}{\Longrightarrow} -b \leq -a \quad \boxtimes$$

(2) Assume $a \leq b$ and $c \leq 0$

(*) $c \leq 0 \overset{(1)}{\Rightarrow} -0 \leq -c \overset{\text{check}}{\Rightarrow} 0 \leq -c$

(**) $a \leq b \overset{O5}{\Rightarrow} a(-c) \leq b(-c) \underset{M2*2}{\overset{(iii)*2}{\Rightarrow}}$

$-ca \leq -cb \overset{(1)}{\Rightarrow} -(-cb) \leq -(-ca)$

$\overset{\text{check}}{\Rightarrow} cb \leq ca \overset{M2*2}{\Rightarrow} bc \leq ac$ $\boxtimes$

(3) $0 \leq a$ and $0 \leq b$:

$0 \leq a \overset{O5}{\Rightarrow} 0 \cdot b \leq a \cdot b \underset{M2}{\overset{(ii)}{\Rightarrow}} 0 \leq ab$ $\boxtimes$

(4) Prove on your own

(5) —— " ——

(6) Suppose $0 < a \Rightarrow a \neq 0$ so by M4, there exist a^{-1} s.t. $aa^{-1} = 1$. Suppose that $a^{-1} \leq 0 \overset{(1)}{\Rightarrow} 0 \leq -a^{-1}$

$\therefore$ $0 < a$ and $0 \leq -a^{-1} \overset{(3)}{\Rightarrow} 0 \leq a(-a^{-1})$

$\overset{(iii)}{=} -(aa^{-1}) \overset{M4}{=} -1 \Rightarrow 0 \leq -1 \overset{(1)}{\Rightarrow}$

$-(-1) \leq -0 \overset{\text{check}}{\Rightarrow} 1 \leq 0$ – contradicts (5)

$\Rightarrow$ our assumption that $a^{-1} \leq 0$ is false.

$\Rightarrow a^{-1} > 0$ $\boxtimes$