Def: The absolute value of x: Given any $x \in \mathbb{R}$,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Def: The distance between $a, b \in \mathbb{R}$ is

$$\text{dist}(a,b) = |b-a|$$

Properties of the absolute value:

Theorem: Suppose that $a, b, c \in \mathbb{R}$, then

(1) $|a| \geq 0$

(2) $|ab| = |a||b|$

(3) $|a+b| \leq |a| + |b|$ – triangle inequality.

Proof: (1) $|a| = \begin{cases} a & \text{if } a \geq 0 \quad \text{– done} \\ -a & \text{if } a < 0 \end{cases}$

When $a < 0 \overset{(1) \text{ in Thm } 3.2}{\Longrightarrow} 0 < -a$ – done $\boxtimes$

(2) $|ab| = \begin{cases} ab & \text{if } ab \geq 0 \\ -ab & \text{if } ab < 0 \end{cases}$

(i) $a \geq 0,\ b \geq 0 \underset{(iii)}{\overset{\text{Thm } 3.2}{\Longrightarrow}} ab \geq 0 \Rightarrow |ab| = ab$

$|a| = a,\ |b| = b$

$\Rightarrow |ab| = ab = |a||b|$ – done

(ii) $a \leq 0,\ b \geq 0 \Rightarrow |a| = -a,\ |b| = b$; Consider ab:

$a \leq 0 \overset{O5}{\Longrightarrow} ab \leq 0 \cdot b \underset{(ii)}{\overset{\text{Thm } 3.1}{\Longrightarrow}} ab \leq 0 \Rightarrow |ab| = -ab$

$\overset{\text{Thm } 3.1 (iii)}{\Longrightarrow} |ab| = (-a)b = |a||b|$ – done

(iii) $a \geq 0$, $b \leq 0$ – same as in (ii)

(iv) $a < 0$, $b < 0$ $\Rightarrow$ $|a| = -a$, $|b| = -b$; Consider ab:

$$a < 0 \underset{(ii)}{\overset{\text{Thm 3.2}}{\Longrightarrow}} 0 \cdot b < ab \underset{(ii)}{\overset{\text{Thm 3.1}}{\Longrightarrow}} 0 < ab \Rightarrow |ab| = ab$$

$$\underset{(iv)}{\overset{\text{Thm 3.1}}{\Longrightarrow}} |ab| = (-a)(-b) = |a||b| \text{ – done} \quad \boxtimes$$

(3) Note that $-|a| \leq a \leq |a|$ – verify this!

$-|b| \leq b \leq |b|$

$\Rightarrow$ $-|a| \leq a \leq |a|$ – add b to all sides:

$$-|a| + b \leq a + b \leq |a| + b$$

also: $b \leq |b|$ – add $|a|$ to both sides: $|a| + b \leq |a| + |b|$

$-|b| \leq b$ – add $-|a|$ to both sides: $-|a| - |b| \leq -|a| + b$

$$\Rightarrow -|a| - |b| \leq -|a| + b \leq a + b \leq |a| + b \leq |a| + |b|$$

$$\Rightarrow -(|a| + |b|) \leq a + b \leq |a| + |b|$$

$$\Rightarrow |a+b| = \begin{cases} a+b \text{ if } a+b \geq 0 & \leq |a| + |b| \\ -(a+b) \text{ if } a+b < 0 & \leq |a| + |b| \end{cases}$$

because $-(|a| + |b|) \leq a + b \Rightarrow -(a+b) \leq |a| + |b|$ $\boxtimes$

$$\text{dist}(a,c) = |c-a| = |c-b+b-a| \leq |c-b| + |b-a|$$

$$= \text{dist}(b,c) + \text{dist}(a,b)$$

$$\Rightarrow \quad \mathrm{dist}(a,c) \leq \mathrm{dist}(b,c) + \mathrm{dist}(a,b)$$

In 2D: