Let $\mathbb{V}$ again be either $\mathbb{Q}$ or $\mathbb{R}$.

<u>Def</u>: An $M \in \mathbb{V}$ is an upper bound for $S \subset \mathbb{V}$ if $s \leq M$ for every $s \in S$

<u>Def</u>: An $m \in \mathbb{V}$ is a lower bound for $S \subset \mathbb{V}$ if $s \geq m$ for every $s \in S$

<u>Note:</u> An upper bound is <u>not</u> unique - if M is an upper bound, then any # larger than M is an upper bound. Similar for lower bounds.

Examples: (1) $\mathbb{V} = \mathbb{R}$, $S = \{1, 2, 3\}$ - $3, 4, \pi$ - examples of upper bounds; $-101, 0, 1$ - examples of lower bounds.

(2) $\mathbb{V} = \mathbb{R}$, $S = (0, 1]$ - $1, 2, 1012$ - examples of upper bounds; $0, -0.1, -53$ - examples of lower bounds.

(3) $\mathbb{V} = \mathbb{R}$, $S = \{x \in \mathbb{R} : 0 \leq x^2 < 2\}$ - any number that is $\geq \sqrt{2}$ is an upper bound; any number ≤ 0 is a lower bound.

(4) $\mathbb{V} = \mathbb{Q}$, $S = \{x \in \mathbb{Q} : 0 \leq x^2 < 2\}$ - any rational # square of which is > 2 is an upper bound ($\sqrt{2}$ is <u>not</u> an upper bound because $\sqrt{2}$ is not in $\mathbb{Q}$!; any rational number ≤ 0 is a lower bound

(5) $V = \mathbb{Q}$, $S = \mathbb{N}$: any rational $\# \leq 1$ is a lower bound; no upper bound.

<u>Def:</u> A set $S \subset V$ is bounded from above if it has an upper bound.

<u>Def:</u> A set $S \subset V$ is bounded from below if it has a lower bound.

<u>Def:</u> A set $S \subset V$ is bounded if it is bounded both from above and below.

Examples: (1) $V = \mathbb{R}$, $S = \{1, 2, 3\}$ - bounded from above and below $\Rightarrow$ S is bounded.

(2) $V = \mathbb{R}$, $S = (0, 1]$ - bounded

(3) $V = \mathbb{R}$, $S = \{x \in \mathbb{R} : 0 \leq x^2 < 2\}$ - bounded

(4) $V = \mathbb{Q}$, $S = \{x \in \mathbb{Q} : 0 \leq x^2 < 2\}$ - bounded

(5) $V = \mathbb{Q}$, $S = \mathbb{N}$ - <u>not</u> bounded, bounded from below.

<u>Def:</u> Suppose that $S \subset V$. The element $s \in S$ is a maximum of S if s is also an upper bound for S. Notation: $\max S$

Def: Suppose that $S \subset V$. The element $s \in S$ is a minimum of S if s is also a lower bound for S. Notation: $\min S$

Examples: (1) $V = \mathbb{R}$, $S = \{1, 2, 3\}$: $\min S = 1$, $\max S = 3$

Note: any set of finitely many #' has both a maximum and a minimum.

(2) $V = \mathbb{R}$, $S = (0, 1]$ - no min, $\max S = 1$.

(3) $V = \mathbb{R}$, $S = \{x \in \mathbb{R} : 0 \leq x^2 < 2\}$ - no max S, $\min S = 0$.

(4) $V = \mathbb{Q}$, $S = \{x \in \mathbb{Q} : 0 \leq x^2 < 2\}$ - no max S, $\min S = 0$.

(5) $V = \mathbb{Q}$, $S = \mathbb{N}$ - no max S, $\min S = 1$

(6) $V = \mathbb{Q}$ $S = \{x \in \mathbb{Q} : 0 < x^2 \leq 2\}$ - no min S; no max S.

Def: The least upper bound for S (if exists) is called the supremum of S and is denoted by $\sup S$.

Def: The greatest lower bound for S (if exists) is called the infimum of S and is denoted by $\inf S$.

(1) $V = \mathbb{R}$, $S = \{1, 2, 3\} \Rightarrow \sup S = 3$
$\inf S = 1$

(2) $V = \mathbb{R}$, $S = (0, 1]$ $\sup S = 1$; $\inf S = 0$

(3) $V = \mathbb{R}$, $S = \{x \in \mathbb{R} : 0 \leq x^2 < 2\}$ - $\inf S = 0$
$\sup S = \sqrt{2}$

(4) $V = \mathbb{Q}$, $S = \{x \in \mathbb{Q} : 0 \leq x^2 < 2\}$ - $\inf S = 0$
$\sup S$ - does not exist.

(5) $V = \mathbb{Q}$, $S = \mathbb{N}$: $\inf S = 1$; $\sup S$ - does not exist.

Suppose that $s_0 = \sup S \iff$ (1) s_0 is an upper bound on S; (2) s_0 is the least upper bound on $S \iff$ (1) $s \leq s_0$ for every $s \in S$; (2) consider any $t < s_0$, then t is not an upper bound S, i.e., for any $t < s_0$, there exists an $s \in S$ s.t. $s > t$.

$\Rightarrow$ To test whether $s_0 = \sup S$ check:

(1) $s \leq s_0$ for all $s \in S$

(2) For any $t < s_0$, there exists an $s \in S$ s.t. $s > t$.

To test whether $s_0 = \inf S$ check:

(1) $s \geq s_0$ for all $s \in S$

(2) For any $t > s_0$, there exists an $s \in S$ s.t. $s < t$.

Example: Show that, if $V = \mathbb{R}$ and $S = \{x \in \mathbb{R} : 0 \leq x^2 < 2\}$, then $\sup S = \sqrt{2}$

<u>Proof</u>: (1) From the definition of S, if $x \in S \Rightarrow x^2 < 2 \Rightarrow x < \sqrt{2}$ $\therefore \sqrt{2}$ is an upper bound on S

(2) Choose any $t < \sqrt{2} \Rightarrow t + \sqrt{2} < \sqrt{2} + \sqrt{2}$

$$\Rightarrow \frac{1}{2}(t + \sqrt{2}) < \frac{1}{2}(\sqrt{2} + \sqrt{2}) = \frac{1}{2} \cdot 2\sqrt{2} = \sqrt{2}$$

Go back to $t < \sqrt{2} \Rightarrow t + t < \sqrt{2} + t$

$$\Rightarrow \frac{1}{2}(t+t) < \frac{1}{2}(\sqrt{2} + t) \Rightarrow$$

$$t < \frac{1}{2}(t + \sqrt{2})$$

$$\therefore \quad t < \frac{1}{2}(t + \sqrt{2}) < \sqrt{2} \underset{t>0}{\Rightarrow} \frac{1}{2}(t+\sqrt{2}) \in S$$

$\Rightarrow \frac{1}{2}(t + \sqrt{2})$ is an element of S that is larger that t. $\boxtimes$