Recall: Consider two sets

$$S = \{x \in \mathbb{R} : 0 \le x, x^2 \le 2\} \subset \mathbb{R}$$

$$T = \{x \in \mathbb{Q} : 0 \le x, x^2 \le 2\} \subset \mathbb{Q}$$

Prove that $\inf S = 0$, $\sup S = \sqrt{2}$

(a) $\inf S = 0$: (1) By definition of S, $x \in S \Rightarrow$ $x \geq 0 \Rightarrow 0$ is a lower bound for S

(2) Choose any small $t > 0 \Rightarrow$

$$\underbrace{0 < \frac{0+t}{2} < t}_{\text{(proved last time)}} < \sqrt{2} \Rightarrow t/2 \in S$$

and t cannot be a lower bound for S since there is an element of S that is smaller than t

$\Rightarrow 0$ is the greatest lower bound for S

(b) $\sup S = \sqrt{2}$: (1) By definition of S, $x \in S \Rightarrow$ $x < \sqrt{2} \Rightarrow \sqrt{2}$ is an upper bound for S

(2) Choose any small $t > 0 \Rightarrow$

$$\sqrt{2} - t \underbrace{< \sqrt{2} - \frac{t}{2} <}_{\text{(proved last time)}} \sqrt{2} \Rightarrow \sqrt{2} - t/2 \in S$$

and $\sqrt{2}-t$ cannot be an upperbound for S since there is an element of S that is larger than $\sqrt{2}-t$

$\Rightarrow \sqrt{2}$ is the least upper bound for S.

The same proof as in (a) can be used to show that the $\inf T = 0$, however, the proof in (b) won't work for T since $\sqrt{2} \notin \mathbb{Q}$. Indeed, suppose that $\sup T = s \in \mathbb{Q} \Rightarrow$ (1) If $s^2 < 2$, then we will show soon that there exists a rational q s.t. $s^2 < q^2 < 2 \Rightarrow q \in T$ and s cannot be an upper bound - contradicts our assumption that s is the supremum of T. (2) If $s^2 > 2$, then there exists a rational q s.t. $2 < q^2 < s^2 \Rightarrow q$ is an upper bound for $T \Rightarrow$ s cannot be the least upper bound for T - contradiction. Then, the only possible remaining choice is that $s \in \mathbb{Q}$ satisfies $s^2 = 2$ - impossible $\Rightarrow \sup T$ does not exist!

Ex: Show that for $T = \{s \in \mathbb{Q} : 0 \leq x, x^2 \leq 4\}$, $\sup S = 2$, $\inf S = 0$

From these examples, we conjecture that for a bounded set in $\mathbb{Q}$, its supremum may or may not exist while for a bounded set in $\mathbb{R}$ the supremum should exist. Thus the difference between $\mathbb{Q}$ and $\mathbb{R}$ can be expressed in the following:

Completeness Axiom: Suppose that a nonempty set $S \subset \mathbb{R}$ is bounded from above. Then $\sup S$ exists.

As we showed above, this axiom is not valid in $\mathbb{Q}$.

Theorem: Suppose that S is a nonempty subset of $\mathbb{R}$ that is bounded from below. Then $\inf S$ exists.

Proof: Clearly we need to use Completeness Axiom.
Define $-S = \{x \in \mathbb{R} : -x \in S\}$. Then $x \in S$ is equivalent to $-x \in -S$. First, we show that $-S$ is bounded from above. Since S is bounded from below, there exists an $M \in \mathbb{R}$ s.t. $x \geq M$ for every $x \in S$ $\Rightarrow$ $-x \leq -M$ for every $x \in S$ $\Leftrightarrow$ $-x \leq -M$ for every $-x \in -S$ $\Rightarrow$ $-M$ is an upper bound on $-S$. By Completeness Axiom, we conclude that $\sup(-S)$ exists. Set $s_0 = \sup(-S)$. Now choose any $x \in S \Rightarrow -x \in -S$ $\Rightarrow -x \leq s_0$ or $x \geq -s_0$ $\Rightarrow$ $-s_0$ is a lower bound on S. Next, choose any $t > -s_0 \Rightarrow s_0 > -t$. Recall that s_0 is the least upper bound on

$-S \Rightarrow$ since $s_0 > -t$, $-t$ is not an upper bound on $-S \Rightarrow$ there exists a $-s \in -S$ s.t. $-s > -t$. Multiplying both sides of the last inequality by -1, we obtain that there is an $s \in S$ s.t. $s < t \Rightarrow t$ is not a lower bound on $S \Rightarrow -s_0$ is the greatest lower bound on $S \Rightarrow \inf S = -s_0 = -\sup(-S)$ ☒

<u>Theorem:</u> (Archimedean Property) If $a > 0$ and $b > 0$, then there exists an $n \in \mathbb{N}$ s.t. $an > b$

<u>Proof:</u> We use contradiction: suppose that AP does not hold $\Rightarrow an \leq b$ for all $n \in \mathbb{N}$

Define a set $S = \{x = an \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Because this set is bounded by b from above, S has a supremum: $s_0 = \sup S$. Because s_0 is the least upper bound on S, then $s_0 - a$ is not an upper bound on $S \Rightarrow$ there exists an element $s \in S$ s.t. $s > s_0 - a$. But all elements of S are positive integer multiples of a, hence there is an $n \in \mathbb{N}$ s.t. $s = na \Rightarrow na > s_0 - a$ or $na + a > s_0 \Rightarrow (n+1)a > s_0$. However, $(n+1)a \in S \Rightarrow$ there exists an element of S that is greater than the supremum of S - contradiction. ☒