Completeness AXIOM $\Rightarrow$ every bounded set $S \subset \mathbb{R}$ has both sup and inf. What about unbounded sets? Introduce the definitions:

Def 1: If S is not bounded from above then we say that $\sup S = \infty$.

Def : If S is not bounded from below then we say that $\inf S = -\infty$

In other words: (1) $\sup S = \infty$ $\Leftrightarrow$ S is not bounded from above $\Leftrightarrow$ S does not have an upper bound $\Leftrightarrow$ $\forall M > 0$, this M is not an upper bound on S $\Leftrightarrow$ $\forall M > 0$, $\exists s \in S$ s.t. $s > M$.

$\therefore$ $\sup S = \infty$ if, for any $M > 0$, there exists $s \in S$ s.t. $s > M$.

(2) $\inf S = -\infty$ $\Leftrightarrow$ S is not bounded from below $\Leftrightarrow$ S does not have a lower bound $\Leftrightarrow$ $\forall M < 0$, this M is not a lower bound on S $\Leftrightarrow$ $\forall M < 0$, $\exists s \in S$ s.t. $s < M$.

$\therefore$ $\inf S = -\infty$ if, for any $M < 0$ there exists $s \in S$ s.t. $s < M$.

Now choose any set $S \subset \mathbb{R} \Rightarrow$ (1) if S is bounded from above $\Rightarrow \sup S =$ real #; (2) if S is not bounded from above $\Rightarrow \sup S = \infty$

(3) if S is bounded from below $\Rightarrow \inf S =$ real #; (4) if S is not bounded from below $\Rightarrow \inf S = -\infty$

$\therefore$ Any set of real #'s has a supremum and infimum.

(1) Suppose $\sup S = -\infty \Rightarrow S = \emptyset$

(2) Suppose $\inf S = \infty \Rightarrow S = \emptyset$

(3) Show that $\sup S = \infty$, where $S = \mathbb{N}$.

Proof: Choose any $M > 0$ and consider M and $1 \overset{A.P.}{\Longrightarrow} \exists n \in \mathbb{N}$ s.t. $n \cdot 1 > M \Rightarrow n > M$ $\therefore$ there exists $n \in S$ s.t. $n > M$ $\Rightarrow \sup S = \infty$

(4) Show that infimum of the set

$S = \{-n^3 : n \in \mathbb{N}\}$ is $-\infty$.

Proof: Recall, $\inf S = -\infty$ if $\forall M < 0$, there exists $s \in S$ s.t. $s < M$. Choose any $M<0$.

$\Rightarrow$ Consider $-M$ and 1 $\Rightarrow$ both $-M$ and $1 > 0 \overset{A.P.}{\Rightarrow}$ there exists an $n \in \mathbb{N}$ s.t. $n \cdot 1 > -M$ or $n > -M$. Since $n \geq 1 \Rightarrow n^2 \geq n \geq 1 \Rightarrow n^2 \geq 1$ $\Rightarrow n^3 \geq n$. Combine this with $n > -M \Rightarrow n^3 > -M$ $\Rightarrow -(n^3) < M \Rightarrow (-n)^3 < M$ $\boxtimes$

(5) Show that for $S = \{\sqrt{n} : n \in \mathbb{N}\}$,

$$\inf S = 1, \quad \sup S = \infty$$

Proof: $\inf S = 1$: (a) By definition of S: $\sqrt{n} \geq 1$ for any $n \in \mathbb{N} \Rightarrow 1$ is a lower bound on S.

(b) Choose any $t > 1 \Rightarrow$ note that $1 \in S \Rightarrow$ there is an element of S below $t \Rightarrow t$ is not a lower bound for S. $\boxtimes$

(b) $\sup S = \infty$: Choose any $M > 0 \underset{\text{for } 1 \text{ and } M^2}{\overset{A.P}{\Rightarrow}} \exists n \in \mathbb{N}$ s.t.

$n \cdot 1 > M^2 \Rightarrow \sqrt{n} > \sqrt{M^2} = |M| \doteq M \Rightarrow$ there exists an element of S larger than M. $\boxtimes$

(6) Suppose $S \subset T$. Then $\sup S \leq \sup T$ and $\inf S \geq \inf T$.

<u>Proof:</u> For every $t \in T$, $t \leq \sup T$ since $\sup T$ is an upper bound on T $\Rightarrow$ because $S \subset T$, we have $s \leq \sup T$ for every $s \in S$ $\Rightarrow$ $\sup T$ is an upper bound on S. At the same time, $\sup S$ is the least upper bound on S $\Rightarrow$ $\sup S \leq \sup T$. Proof of $\inf S \geq \inf T$ is similar.

(7) Suppose that the sets A and B are nonempty. Define $A+B = \{s+t : s \in A, t \in B\}$. Then

$$\sup(A+B) = \sup A + \sup B$$

<u>Proof:</u> Note that $A, B \neq \emptyset \Rightarrow \sup A, \sup B \neq -\infty$

(a) Suppose that at least one of the sets A & B is not bounded from above; w.l.o.g. assume that A is not bounded from above $\Rightarrow \sup A = \infty$

Since $\infty + \underset{\text{any number}}{a} = \infty + \infty = \infty \Rightarrow \sup A + \sup B = \infty$

Now, choose any $b \in B$ and define $A+b = \{s+b : s \in A\}$ $\Rightarrow$ $A+b$ is not bounded from above (prove this!), hence $\sup(A+b) = \infty$. However,

$$A+b \subset A+B \overset{(6)}{\Rightarrow} \sup(A+b) \leq \sup(A+B) \Rightarrow \sup(A+B) = \infty \quad \boxtimes$$

(b) Now suppose that both A and B are bounded from above $\Rightarrow$ $\sup A$ and $\sup B$ are finite #'s and $s \leq \sup A$ for any $s \in A$ and $t \leq \sup B$ for all $t \in B$ $\Rightarrow$ For any $s \in A$ and $t \in B$,

$$s+t \leq \sup A + \sup B$$

that is $\sup A + \sup B$ is an upper bound on $A+B$. It remains to show that $\sup A + \sup B$ is the least upper bound on $A+B$. To demonstrate this, note that if $\varepsilon > 0$, then

$\sup A - \frac{\varepsilon}{2}$ is not an upper bound on A $\Rightarrow$ there exists an $s_0 \in A$ s.t. $s_0 > \sup A - \frac{\varepsilon}{2}$

$\sup B - \frac{\varepsilon}{2}$ is not an upper bound on B $\Rightarrow$ there exists an $t_0 \in B$ s.t. $t_0 > \sup B - \frac{\varepsilon}{2}$

$$\Rightarrow s_0 + t_0 \in A+B \text{ and } s_0 + t_0 > \sup A - \frac{\varepsilon}{2} + \sup B - \frac{\varepsilon}{2}$$

$$\Rightarrow s_0 + t_0 > \sup A + \sup B - \varepsilon$$

Conclusion: $\sup A + \sup B - \varepsilon$ is <u>not</u> an upper bound on $A+B$ for any $\varepsilon > 0$.

$$\Rightarrow \sup(A+B) = \sup A + \sup B \quad \boxtimes$$