

Pb 6: Suppose that $\lim_{n \rightarrow \infty} s_n = 1$. Given

an $\varepsilon > 0$, there exists a $N \in \mathbb{R}$ s.t.

$|s_n - 1| < \frac{\varepsilon}{3}$ if $n > N$. Then, if $n > N$,

$$|(3s_n + 1) - 4| = |3s_n - 3| = 3|s_n - 1| < 3 \frac{\varepsilon}{3} = \varepsilon.$$

We conclude that $\lim_{n \rightarrow \infty} (3s_n + 1) = 4$.

Pb 7: see the book

Pb 8: $\sup S$ exists by the Completeness

Axiom. Conjecture: $\sup S = \sqrt{3}$

Proof: For every $a \in (0, \sqrt{3})$, $a < \sqrt{3}$

$\Rightarrow \sqrt{3}$ is an upper bound on S .

Given $\varepsilon > 0$, suppose that $\sqrt{3} - \varepsilon$ is also an upper bound; however, by denseness of $\mathbb{Q}$

there is $\sqrt{3} - \varepsilon < a < \sqrt{3}$ $a \in \mathbb{Q}$. Thus $a \in S$ and

$\sqrt{3} - \varepsilon < a$ - contradiction. We conclude that

$\sqrt{3}$ is the least upper bound.

Pb. 9: Suppose that $\sup S$ exists in $\mathbb{Q}$ and is equal to a . Then $a \in \mathbb{Q}$ and thus in $\mathbb{R}$. If S is considered as a subset of $\mathbb{R}$, then $\sup S = \sqrt{3}$ (argue as in Pb. 8). Thus $a = \sqrt{3} \notin \mathbb{Q}$ - contradiction
 $\therefore S$ has no supremum in $\mathbb{Q}$

Pb 10: Given any $a \in S$ (exists since S is nonempty, $\inf S \leq a$, $a \leq \sup S$ by definitions of $\inf$ and $\sup \Rightarrow$)

$$\inf S \leq a \leq \sup S$$
$$\Rightarrow \inf S \leq \sup S$$

Pb 11: See the book

Pb 12: Let $x = \sqrt{7} \Rightarrow x^2 = 7 \Rightarrow x^2 - 7 = 0$
by rational zeros thm, if x is a rational solution of this equation, then $x = \frac{p}{q}$, where p divides 7 and q

divides 1 $\Rightarrow p = \pm 7, \pm 1, q = \pm 1$

$\Rightarrow x = \pm 7$ or ± 1 ; however

$$(\pm 7)^2 - 7 \neq 0 \text{ and } (\pm 1)^2 - 7 \neq 0$$

contradiction. $\Rightarrow x$ is irrational.

Prob 13: Suppose $\lim_{n \rightarrow \infty} s_n = s \Rightarrow$

For any $\epsilon > 0$, there exists $N \in \mathbb{R}$ s.t.

$n > N \Rightarrow |s_n - s| < \epsilon$. Suppose $n > N$,

then $|t_n - s| = |s_{2n} - s| < \epsilon$, because

$n > N \Rightarrow 2n > N$. $\square$