$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

– n-th order linear ODE.

The corresponding IVP is:

$$\begin{cases} a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

Note: If $x_0 \in (a,b)$ and $a_i(x)$, $i=0,\dots,n$ and $f(x)$ are all continuous functions of x on (a,b) and $a_n(x) \neq 0$ on (a,b), then the IVP has a unique solution on (a,b).

Ex: $\begin{cases} y'' + 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ – The coefficient functions are $a_2(x) = 1$ $a_0(x) = 2$ $a_1(x) = 3$ $f(x) = 0$ – all constants and therefore continuous for all x; $a_2(x) \neq 0 \Rightarrow$ unique solution of IVP exists.

Check that $y(x) = c_1 e^{-x} + c_2 e^{-2x}$ is a solution of this ODE:

$y'(x) = -c_1 e^{-x} - 2c_2 e^{-2x}$; $y''(x) = c_1 e^{-x} + 4c_2 e^{-2x}$

substitute into the ODE: $c_1 e^{-x} + 4c_2 e^{-2x} + 3(-c_1 e^{-x} - 2c_2 e^{-2x}) + 2(c_1 e^{-x} + c_2 e^{-2x}) = \cancel{c_1 e^{-x}} + \cancel{4c_2 e^{-2x}} - \cancel{3c_1 e^{-x}} - \cancel{6c_2 e^{-2x}} + \cancel{2c_1 e^{-x}} + \cancel{2c_2 e^{-2x}} = e^{-x}(\underbrace{c_1 - 3c_1 + 2c_1}_{0}) + e^{-2x}(\underbrace{4c_2 - 6c_2 + 2c_2}_{0}) = 0 \Rightarrow$

$$\begin{cases} 1 = y(0) = c_1 e^{-0} + c_2 e^{-2\cdot 0} \\ 0 = y'(0) = -c_1 e^{-0} - 2c_2 e^{-2\cdot 0} \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ -c_1 - 2c_2 = 0 \end{cases}$$

Add: $0 - c_2 = 1 \Rightarrow c_2 = -1$

$\Rightarrow c_1 = 1 - c_2 = 1 - (-1) = 2$

Solution to IVP $\Leftarrow$

$$y(x) = 2e^{-x} - e^{-2x}$$

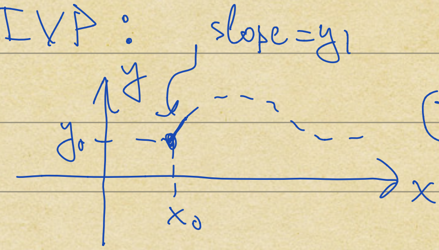
- this is the only solution

Ex: $y'' + \pi^2 y = 0$ - second order linear homogeneous equation

(IVP) $\begin{cases} y'' + \pi^2 y = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$ - always has a unique solution.

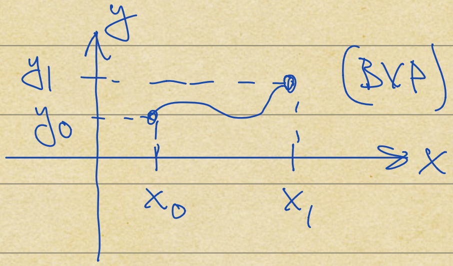
An alternative to an IVP:

$$\begin{cases} y'' + \pi^2 y = 0 \\ y(x_0) = y_0 \\ y(x_1) = y_1 \end{cases}$$



(IVP)

Consider particular examples of BVP:



First, observe that

$y = c_1 \cos \pi x + c_2 \sin \pi x$

is the general solution of $y'' + \pi^2 y = 0$:

$y' = -\pi c_1 \sin \pi x + \pi c_2 \cos \pi x \Rightarrow y'' = -\pi^2 c_1 \cos \pi x - \pi^2 c_2 \sin \pi x$

$= -\pi^2 (c_1 \cos \pi x + c_2 \sin \pi x)$

$= -\pi^2 y \Rightarrow y'' + \pi^2 y = 0$

– checks out.

$$\begin{cases} y'' + \pi^2 y = 0 \\ y(0) = 1 \\ y(1) = 2 \end{cases} \quad (\text{BVP1})$$

$y = c_1 \cos \pi x + c_2 \sin \pi x \Rightarrow 1 = y(0) = c_1 \cos \pi \cdot 0 + c_2 \sin \pi \cdot 0$

$2 = y(1) = c_1 \cos \pi \cdot 1 + c_2 \sin \pi \cdot 1$

$\Rightarrow \quad c_1 = 1 ; \ -c_1 = 2 \Rightarrow c_1 = -2 \Rightarrow$ impossible!

$\Rightarrow$ (BVP1) does not have a solution!!!

$$\begin{cases} y'' + \pi^2 y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases} \quad (\text{BVP2})$$

$0 = y(0) = c_1 \cos \pi \cdot 0 + c_2 \sin \pi \cdot 0$

$0 = y(1) = c_1 \cos \pi \cdot 1 + c_2 \sin \pi \cdot 1$

$\Rightarrow \quad c_1 = 0 ; \ -c_1 = 0$

$c_1 = 0$, c_2 is arbitrary

$\Rightarrow$ A solution is $y = c_2 \sin \pi x$ – ∞ many solutions $\Rightarrow$ (BVP2) has ∞ many solutions!!!

$$\begin{cases} y'' + \pi^2 y = 0 \\ y(1) = 0 \\ y\left(\frac{3}{2}\right) = 0 \end{cases} \quad \text{(BVP3)} \qquad \begin{aligned} 0 &= y(1) = c_1 \cos \pi \cdot 0 + c_2 \sin \pi \cdot 0 \\ 0 &= y\left(\tfrac{3}{2}\right) = c_1 \cos \frac{\pi \cdot 3}{2} + c_2 \sin \frac{\pi \cdot 3}{2} \end{aligned}$$

$$\Downarrow$$

$$y(x) = 0 \quad \Leftarrow \quad \begin{cases} c_1 = 0; \quad 0 = \cancel{c_1} \cdot 0 - c_2 \\ \Rightarrow c_1 = c_2 = 0 \end{cases}$$

- the only solution of (BVP3)

$\therefore$ A BVP may have one solution, ∞ many solutions, or no solutions at all.

Homogeneous Linear Equations:

Consider the second order linear homogeneous ODE:

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0 \qquad (*)$$

1. Suppose that y_1 and y_2 are solutions of $(*)$, that is,

$$a_2(x) y_1'' + a_1(x) y_1' + a_0(x) y_1 = 0$$

and

$$a_2(x)\,y_2'' + a_1(x)\,y_2' + a_0(x)\,y_2 = 0$$

Consider a function

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \text{ where}$$

c_1 and c_2 are arbitrary constants. We say that y is a <u>linear combination</u> of y_1 and y_2

$$y' = (c_1y_1 + c_2y_2)' = c_1y_1' + c_2y_2' \Rightarrow$$

$$y'' = (y')' = (c_1y_1' + c_2y_2')' = c_1y_1'' + c_2y_2''$$

$$\Rightarrow a_2y'' + a_1y' + a_0y = a_2(c_1y_1'' + c_2y_2'') + a_1(c_1y_1' + c_2y_2')$$

$$+ a_0(c_1y_1 + c_2y_2) = a_2c_1y_1'' + a_2c_2y_2'' + a_1c_1y_1' + a_1c_2y_2'$$

$$+ a_0c_1y_1 + a_0c_2y_2 = c_1(\cancelto{0}{a_2y_1'' + a_1y_1' + a_0y_1})$$

$$+ c_2(\cancelto{0}{a_2y_2'' + a_1y_2' + a_0y_2}) = c_1\cdot 0 + c_2\cdot 0 = 0 \Rightarrow$$

$y = c_1y_1 + c_2y_2$ satisfies $a_2y'' + a_1y' + a_0y = 0$

$\Rightarrow$ If y_1 and y_2 solve (*) then any linear combination of y_1 and y_2 solves (*) – superposition principle.

<u>Ex:</u> Consider the second order linear ODE

(#) $y'' + 4y = 0$. Verify that $y_1 = \sin 2x$ and $y_2 = \cos 2x$ are solutions of (#):

$y_1' = 2\cos 2x$, $y_2' = -2\sin 2x$

$y_1'' = -4\sin 2x$, $y_2'' = -4\cos 2x$ $\Rightarrow$ $y_1'' + 4y_1 = -4\sin 2x + 4\sin 2x = 0$

$y_2'' + 4y_2 = -4\cos 2x + 4\cos 2x = 0$

$y = C_1 \cos 2x + C_2 \sin 2x \xleftarrow{\text{superposition pr.}}$

solves (#) for any combination of C_1 and C_2 $\Rightarrow$ y is a two-parameter family of solutions to the second order equation (#) $\Rightarrow$ $y = C_1\cos 2x + C_2 \sin 2x$ – a general solution of (#).

$\therefore$ To find the general solution of the second order ODE, it is enough to find two solutions and form their linear combination.

<u>Ex:</u> $y'' - y = 0$ (1) Show that $y_1(x) = e^{-x}$ and $y_2(x) = e^{x}$ are solutions.

(2) Find the general solution of the ODE.

(1) $y_1' = -e^{-x} \Rightarrow y_1'' = e^{-x}$ $\quad y_1'' - y_1 = e^{-x} - e^{-x} = 0$

$y_2' = y_2'' = e^x$ $\quad \Rightarrow \quad y_2'' - y_2 = e^x - e^x = 0$

(2) $y = c_1 e^{-x} + c_2 e^x$

Ex: $y'' - y = 0$ (1) Show that $y_1(x) = 2e^x$ and $y_2(x) = e^x$ are solutions.

(2) Find the general solution of the ODE.

(1) $y_1' = 2e^x \Rightarrow y_1'' = 2e^x$ $\quad y_1'' - y_1 = 2e^x - 2e^x = 0$

$y_2' = y_2'' = e^x$ $\quad \Rightarrow \quad y_2'' - y_2 = e^x - e^x = 0$

(2) $y = c_1 2e^x + c_2 e^x = (2c_1 + c_2) e^x = ce^x$

– one parameter family of solutions $\Rightarrow$ not a general solution!

What is the difference between the two examples? In the Ex 2, y_2 is a constant multiple of y_1 $\Rightarrow$ the linear combination of y_1 and y_2 is still a constant multiple of y_1.

$\therefore$ To construct a general solution should have y_2 that is not a constant multiple of y_1.

Terminology: If y_2 is a constant multiple of y_1, then y_1 and y_2 are linearly dependent; otherwise, y_1 and y_2 are linearly independent.

$\Rightarrow$ $e^x, 2e^x$ are linearly dependent

e^x, e^{-x} are linearly independent

$\Rightarrow$

$\therefore$ To find the general solution of the second order ODE, it is enough to find two linearly independent solutions of this ODE and form their linear combination.

Generalizing this to n-th order ODE's:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Superposition principle: If $y_1, \dots, y_k$

solve this equation, then a linear combination

$$y_1 = c_1 y_1 + \dots + c_k y_k$$

also solves the same equation.

<u>Ex:</u> $y_1 = x$, $y_2 = \sin^2 x$, $y_3 = \cos^2 x$, $y_4 = 1$

Form a linear combination:

$$y = c_1 x + c_2 \sin^2 x + c_3 \cos^2 x + c_4 \cdot 1$$

- this involves 4 generic constants. However,

$$\sin^2 x + \cos^2 x = 1 \quad \Rightarrow$$

$$y = c_1 x + c_2 \sin^2 x + c_3 \cos^2 x + c_4 (\sin^2 x + \cos^2 x)$$

$$= c_1 x + \underbrace{(c_2 + c_4)}_{\bar{c}_2} \sin^2 x + \underbrace{(c_3 + c_4)}_{\bar{c}_3} \cos^2 x$$

$$= c_1 x + \bar{c}_2 \sin^2 x + \bar{c}_3 \cos^2 x$$ – really, there are only 3 unknown constants.

<u>Def:</u> $\{y_1, \dots, y_k\}$ are linearly dependent if one of these function can be written as a linear combination of some other functions

from this list. Otherwise $\{y_1, \dots, y_k\}$ are linearly independent.

Ex: $y_1 = x$, $y_2 = \sin^2 x$, $y_3 = \cos^2 x$, $y_4 = 1$

– linearly dependent because

$$y_4 = y_2 + y_3$$

Ex: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ – linearly independent.

Therefore, to find the general solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

need to find n linearly independent solutions of this equation and form their linear combination.

Matrices and determinants:

(1) An $n \times m$ table of #'s is an $n \times m$ matrix:

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ – 2×2 matrix; $B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 6 \end{pmatrix}$ – 2×3 matrix

$D = \begin{pmatrix} -1 & 1 & 6 \\ 0 & 1 & -1 \\ 2 & 3 & 5 \end{pmatrix}$ – 3×3 matrix

If $n=m$ – square matrix $\Rightarrow$ A, D above are square matrices.

(2) Given a square matrix, we can define its determinant:

(I) If A is a 2x2 matrix: $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$

$$\Rightarrow \det A = a_{11}a_{22}-a_{12}a_{21}$$

For example: (a) $\det\begin{pmatrix} 2 & -1 \\ 1 & 3\end{pmatrix} = 2\cdot 3-(-1)\cdot 1 = 7$

(b) $\begin{vmatrix} 1 & 2 \\ -1 & 6\end{vmatrix} = 6\cdot 1-(-1)\cdot 2 = 8$

(c) Find the determinant of $A=\begin{pmatrix} 5 & 0 \\ 1 & 6\end{pmatrix}$:

$\det A = 5\cdot 6 - 1\cdot 0 = 30$

$$\begin{vmatrix} 5 & 0 \\ 1 & 6\end{vmatrix} = 5\cdot 6-1\cdot 0 = 30$$

(II) If A is a 3x3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$$

Ex. (a) Find $\det A$ if

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix} \Rightarrow \det A = 1\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 1(-1\cdot 1 - 1\cdot 3) + 2(1\cdot 3 - (-1)\cdot 2)$$

$$= 1(-4) + 2(3+2) = 6$$

(b) $$\begin{vmatrix} 5 & 1 & 2 \\ 1 & 0 & 0 \\ 3 & 4 & -5 \end{vmatrix} = 5\begin{vmatrix} 0 & 0 \\ 4 & -5 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 3 & -5 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$$

$$= 5\cdot 0 - 1(1\cdot(-5) - 0\cdot 3) + 2(1\cdot 4 - 3\cdot 0) = 13.$$

Back to ODEs:

Suppose that y_1, y_2, y_3 are solutions of

$$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Then $\{y_1, y_2, y_3\}$ is linearly independent if

$$\underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{W(y_1, y_2, y_3)} \neq 0 \text{ for all values of } x.$$

$\Rightarrow$ $y = c_1y_1 + c_2y_2 + c_3y_3$ is the general solution of the ODE.

Ex: $y'' + y = 0$. Show that $y_1 = \sin x$, $y_2 = \cos x$ are linearly independent solutions of the ODE:

(a) $y_1' = \cos x \quad y_2' = -\sin x$
$y_1'' = -\sin x \quad y_2'' = -\cos x$
$\Rightarrow \quad y_1'' + y_1 = -\sin x + \sin x = 0$
$y_2'' + y_2 = -\cos x + \cos x = 0$

$\therefore$ y_1 and y_2 solve the ODE.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin x \cdot \sin x$$
$$- \cos x \cdot \cos x = -(\sin^2 x + \cos^2 x) = -1 \neq 0$$

$\Rightarrow$ y_1 and y_2 are linearly independent $\Rightarrow$

$y(x) = c_1 \sin x + c_2 \cos x$ is a general solution of the ODE.

$\Rightarrow$ $\sin x$, $\cos x$ form a fundamental set of solutions

Generally, any n linearly independent solutions of an n-th order linear homogeneous ODE form a fundamental set of solutions of the ODE.

Ex. Show that $y_1 = e^x$, $y_2 = \sin x$, $y_3 = \cos x$ form a fundamental set of solutions for

$$y''' - y'' + y' - y = 0 \qquad (*)$$

(1) Verify that y_1, y_2, y_3 solve (*):

$y_1 = e^x$, $y_1' = e^x$, $y_1'' = e^x$, $y_1''' = e^x$:

$$y_1''' - y_1'' + y_1' - y_1 = e^x - e^x + e^x - e^x = 0$$

$\Rightarrow$ y_1 solves (*)

$y_2 = \sin x$, $y_2' = \cos x$, $y_2'' = -\sin x$, $y_2''' = -\cos x$

$$y_2''' - y_2'' + y_2' - y_2 = -\cancel{\cos x} - (\cancel{-\sin x}) + \cancel{\cos x} - \cancel{\sin x} = 0$$

$\Rightarrow$ y_2 solves (*)

$y_3 = \cos x$, $y_3' = -\sin x$, $y_3'' = -\cos x$, $y_3''' = \sin x$

$$y_3''' - y_3'' + y_3' - y_3 = \cancel{\sin x} - (\cancel{-\cos x}) - \cancel{\sin x} - \cancel{\cos x} = 0$$

$\Rightarrow$ y_3 solves (*)

(2) Verify that y_1, y_2, y_3 are linearly independent:

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & \sin x & \cos x \\ e^x & \cos x & -\sin x \\ e^x & -\sin x & -\cos x \end{vmatrix}$$

$$= e^x \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} - \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix}$$

$$+ \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} = e^x(-\cos^2 x - \sin^2 x)$$

$$-\sin x\left(-\cos x e^{x}+\sin x e^{x}\right)+\cos x\left(-e^{x}\sin x-e^{x}\cos x\right)$$

$$=e^{x}\left[-1+\cancel{\sin x\cos x}-\sin^{2}x-\cancel{\sin x\cos x}-\cos^{2}x\right]$$

$$=-2e^{x}\neq 0 \Rightarrow y_1, y_2, y_3 \text{ are lin. independent}$$

<u>done!</u>

Nonhomogeneous linear ODE's (use the example of second order). Consider:

$$a_2(x)y''+a_1(x)y'+a_0(x)y=0 \qquad (I)$$

and

$$a_2(x)y''+a_1(x)y'+a_0(x)y=f(x) \qquad (II)$$

Suppose that y_h satisfies (I) and y_p satisfies (II). What can we say about $y=y_p+y_h$?

$$\Rightarrow a_2(x)(y_p+y_h)''+a_1(x)(y_p+y_h)'+a_0(x)(y_p+y_h)$$

$$=a_2(x)(y_p''+y_h'')+a_1(x)(y_p'+y_h')+a_0(x)(y_p+y_h)$$

$$=a_2y_p''+a_2y_h''+a_1y_p'+a_1y_h'+a_0y_p+a_0y_h$$

$$=\underbrace{\left(a_2y_p''+a_1y_p'+a_0y_p\right)}_{f(x)}+\underbrace{\left(a_2y_h''+a_1y_h'+a_0y_h\right)}_{=0}=f(x)$$

$\Rightarrow$ $a_2(x)(y_p+y_h)'' + a_1(x)(y_p+y_h)' + a_0(x)(y_p+y_h) = f(x)$

$\Rightarrow$ $y_p + y_h$ is a solution of (II). $\therefore$ adding any solution of (I) to a solution of (II) produces a solution of (II). – superposition principle for nonhomogeneous ODE.

Because y_h is any solution of (I), as y_h we can use the general solution of (I) $y = c_1y_1 + c_2y_2$

$\Rightarrow$ $y = y_p + c_1y_1 + c_2y_2$ solves the equation (II) and contains two unknowns c_1, and c_2.

$\Rightarrow$ y is the general solution of (II).