Given $y'' + y = 0$, suppose that we know that $y_1 = \cos x$ is a solution of this equation. Indeed, $y_1' = -\sin x$, $y_1'' = -\cos x \Rightarrow y_1'' + y_1 = -\cos x + \cos x = 0$. We want to find a second, linearly independent solution. Look for this solution in the form $y_2(x) = v(x) y_1(x) = v(x)\cos x$. Need to find an ODE for v: $y_2' = v'\cos x + v(-\sin x)$

$\Rightarrow y_2' = v'\cos x - v\sin x \Rightarrow y_2'' = v''\cos x + v'(-\sin x)$
$- v'\sin x - v\cos x = v''\cos x - 2v'\sin x - v\cos x$

Since want $y_2'' + y_2 = 0 \Rightarrow v''\cos x - 2v'\sin x - \cancel{v\cos x}$
$+ \cancel{v\cos x} = 0 \Rightarrow v''\cos x - 2v'\sin x = 0$

– linear second order homogeneous ODE.

Now set $u = v' \Rightarrow u' = v'' \overset{\text{subst.}}{\Rightarrow}$

$$u'\cos x - 2u\sin x = 0$$

– linear 1st order ODE, solve by using the integrating factor:

$$u' - \frac{2\sin x}{\cos x} u = 0$$

Then

$$\mu(x) = e^{-\int \frac{2\sin x}{\cos x}dx} \underset{dp = -\sin x\, dx}{\overset{p=\cos x}{=}} e^{2\int \frac{dp}{p}}$$

$$= e^{2\ln|p|} = e^{\ln p^2} = p^2 = \cos^2 x$$

$$\Rightarrow (\cos^2 x\, u)' = 0 \Rightarrow \cos^2 x\, u = 1 \leftarrow \text{set } C \text{ to } 1.$$

$$\Rightarrow u = \frac{1}{\cos^2 x} \Rightarrow v' = \sec^2 x \Rightarrow v = \int \sec^2 x\, dx$$

$$= \tan x$$

$$\Rightarrow y_2(x) = v(x)\cos x = \tan x \cos x = \frac{\sin x}{\cancel{\cos x}}\cancel{\cos x} = \sin x$$

Clearly $\sin x \neq C\cos x \Rightarrow y_1$ and y_2 are lin. ind.

Check:

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x$$

$$- (-\sin^2 x) = 1 \neq 0$$

$\therefore$ $\sin x$ and $\cos x$ form a fundamental set of $y'' + y = 0 \Rightarrow y(x) = c_1 \cos x + c_2 \sin x$

This method can be reduced to a single formula: Suppose we have a second order linear homogeneous equation in the normal form:

$$y'' + p(x)y' + q(x)y = 0$$

Also suppose that y_1 is a solution of this ODE

$\Rightarrow$ the second linearly independent solution is

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx$$ – reduction of order.

Ex: $y'' + y = 0$, $y_1(x) = \cos x$

$$\Rightarrow y_2 = \cos x \int \frac{e^{-\int 0 dx}}{\cos^2 x} dx = \cos x \int \frac{1}{\cos^2 x} dx$$

$$= \cos x \tan x = \sin x$$

$\Rightarrow$ general solution of $y'' + y = 0$ is

$$y = c_1 \sin x + c_2 \cos x$$

Ex: $x^2 y'' - x y' + y = 0$, $x > 0$

This equation has a solution $y_1 = x$; Indeed $y_1' = 1$, $y_1'' = 0 \Rightarrow$

$$x^2 y_1'' - x y_1' + y_1 = 0 - x \cdot 1 + x = 0$$

Use reduction of order to find the second solution:

$$y'' \underbrace{- \frac{1}{x}}_{p(x)} y' + \frac{1}{x^2} y = 0$$

$$y_2 = y_1 \int \frac{e^{-\int p(x) dx}}{y_1^2} dx$$

$$= x \int \frac{e^{\int \frac{1}{x} dx}}{x^2} dx = x \int \frac{e^{\ln x}}{x^2} dx$$

$$= x \int \frac{x}{x^2} dx = x \int x^{-1} dx$$

$$= x \ln x = y_2$$

Check linear independence:

$$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x(\ln x + 1) - x\ln x$$

$= x \neq 0$ - yes x and $x\ln x$ are linearly

because $x > 0$. independent $\Rightarrow$

General solution: $y = c_1 x + c_2 x \ln x$