$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

- here $a_0, \dots, a_n$ are constants.

$\Rightarrow$ a second order linear homogeneous ODE:

$$a_2 y'' + a_1 y' + a_0 y = 0$$

e.g. $2y'' + 6y' + y = 0$ but not $xy'' + 2y' + x^2 y = 0$

Want to solve

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Try a solution in the form $y = e^{rx}$, where r is an unknown constant.

$$y' = re^{rx}; \; y'' = r^2 e^{rx}$$

$\Rightarrow$ substitute:

$$a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$e^{rx}(a_2 r^2 + a_1 r + a_0) = 0$$

Because $e^{rx} \neq 0$, divide by e^{rx}:

$$a_2 r^2 + a_1 r + a_0 = 0 \text{ - characteristic equation}$$

- quadratic equation for r; no x left!

$\Rightarrow$ solve the characteristic equation:

$$r = \frac{-a_1 \pm (a_1^2 - 4a_0a_2)^{1/2}}{2a_2}$$

3 Choices: (1) $a_1^2 - 4a_0a_2 > 0 \Rightarrow$ two solutions r_1 and r_2 of the characteristic equation $\Rightarrow y_1 = e^{r_1x}$, $y_2 = e^{r_2x}$ solve the ODE; Clearly $e^{r_1x} \neq ce^{r_2x} \Rightarrow y_1$ and y_2 are linearly independent $\Rightarrow$ General solution of the ODE is

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

(2) $a_1^2 - 4a_0a_2 = 0 \Rightarrow r = -\frac{a_1}{2a_2}$ is the only solution of the characteristic equation $\Rightarrow e^{rx}$ is a solution of the ODE. Use the method of reduction of order to find the second solution: Write the ODE in a normal form:

$$y'' + \frac{a_1}{a_2}y' + \frac{a_0}{a_2}y = 0 \Rightarrow$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1(x)^2}dx = e^{rx}\int \frac{e^{-\int \frac{a_1}{a_2}dx}}{(e^{rx})^2}dx$$

$$= e^{rx}\int \frac{e^{-a_1/a_2 x}}{\left(e^{-a_1/2a_2 x}\right)^2}\,dx = e^{rx}\int \frac{\cancel{e^{-\frac{a_1}{a_2}x}}}{\cancel{e^{-\frac{a_1}{a_2}x}}}\,dx = xe^{rx}$$

$\Rightarrow$ $y_1 = e^{rx}$, $y_2 = xe^{rx}$ is a fundamental set of solutions of the ODE $\Rightarrow$

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

(3) $a_1^2 - 4a_0a_2 < 0 \Rightarrow r = \dfrac{-a_1 \pm (a_1^2 - 4a_0a_2)^{1/2}}{2a_2}$

$$= \frac{-a_1 \pm \left(-(4a_0a_2 - a_1^2)\right)^{1/2}}{2a_2} = \frac{-a_1 \pm \sqrt{-1}\,(4a_0a_2 - a_1^2)^{1/2}}{2a_2}$$

$$= \frac{-a_1 \pm i(4a_0a_2 - a_1^2)^{1/2}}{2a_2} = \underbrace{-\frac{a_1}{2a_2}}_{\alpha} \pm \underbrace{\frac{(4a_0a_2 - a_1^2)^{1/2}}{2a_2}}_{\beta}\, i$$

$= \alpha \pm i\beta$ - complex conjugates $\Rightarrow$

General (complex) solution of the ODE

$c_1 e^{(\alpha - i\beta)x} + c_2 e^{(\alpha + i\beta)x}$ - solves the ODE

where c_1 and c_2 are arbitrary complex #'s.

$$c_1 e^{(\alpha - i\beta)x} + c_2 e^{(\alpha + i\beta)x} = c_1 e^{\alpha x - i\beta x} + c_2 e^{\alpha x + i\beta x}$$

$$= c_1 e^{\alpha x} e^{-i\beta x} + c_2 e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(c_1 e^{-i\beta x} + c_2 e^{i\beta x}\right);$$

(a) Choose $c_1 = c_2 = \frac{1}{2} \Rightarrow$

$$e^{\alpha x}\left(\frac{1}{2}e^{-i\beta x} + \frac{1}{2}e^{i\beta x}\right) = e^{\alpha x}\,\frac{e^{i\beta x} + e^{-i\beta x}}{2}$$

$$= e^{\alpha x}\,\frac{\cos(\beta x) + i\sin\beta x + \cos(-\beta x) + i\sin(-\beta x)}{2}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$= e^{\alpha x}\,\frac{\cos\beta x + \cancel{i\sin\beta x} + \cos\beta x - \cancel{i\sin\beta x}}{2}$$

$= e^{\alpha x}\cos\beta x$ – real solution of the ODE

(b) Choose $c_1 = -\frac{1}{2i}$, $c_2 = \frac{1}{2i} \Rightarrow$

$$e^{\alpha x}\left(-\frac{1}{2i}e^{-i\beta x} + \frac{1}{2i}e^{i\beta x}\right) = e^{\alpha x}\,\frac{e^{i\beta x} - e^{-i\beta x}}{2i}$$

$$= e^{\alpha x}\,\frac{\cos(\beta x) + i\sin\beta x - \cos(-\beta x) - i\sin(-\beta x)}{2i}$$

$$= e^{\alpha x}\,\frac{\cancel{\cos\beta x} + i\sin\beta x - \cancel{\cos\beta x} + i\sin\beta x}{2i} = e^{\alpha x}\,\frac{\cancel{2i}\sin\beta x}{\cancel{2i}} = e^{\alpha x}\sin\beta x$$

– real solution of the ODE;

$\Rightarrow$ General real solution of the ODE:

$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

$$\Rightarrow y(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$$