A system of first order linear ODE's is

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + f_1, \\ x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + f_2, \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + f_n, \end{cases}$$

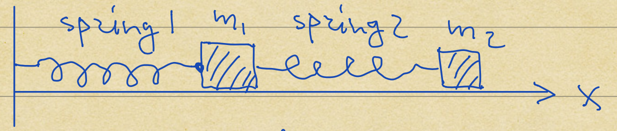
where a_{ij}, $i,j = 1,\dots,n$; f_i, $i = 1,\dots,n$ are given functions of t and x_i, $i=1,\dots,n$ are unknown functions of t.

If $f_1 = f_2 = \dots = f_n \equiv 0$, then the system is homogeneous; otherwise it is nonhomogeneous.

Examples:

(1) $$\begin{cases} x_1' = tx_1 + 2x_2 + t^2x_3 + \sin t \\ x_2' = e^t x_1 + e^{2t}x_2 - 3x_3 + \cos t \\ x_3' = (\cosh t)x_1 - t^3x_2 + x_3 - \sqrt{t} \end{cases}$$ – a 3×3 nonhomogeneous system.

(2) $$\begin{cases} x_1' = 2x_1 + 3x_2 \\ x_2' = -x_1 + x_2 \end{cases}$$ – a 2×2 homogeneous system

(3) 

Elongation of the first spring is $x_1(t)$, elongation of the second spring is $x_2(t)$;

Spring constant of the i-th spring is k_i; $m_1 = m_2 = 0$

friction constants: γ_1, γ_2

Second Newton's Law: $$\begin{cases} \gamma_1 \dot{x}_1 = -k_1x_1 + k_2x_2, \\ \gamma_2 (x_1+x_2)^{\boldsymbol{\cdot}} = -k_2x_2. \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = -\frac{k_1}{\gamma_1}x_1 + \frac{k_2}{\gamma_1}x_2, \\ \dot{x}_2 = -\frac{k_2}{\gamma_2}x_2 - \dot{x}_1. \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = -\frac{k_1}{\gamma_1}x_1 + \frac{k_2}{\gamma_2}x_2, \\ \dot{x}_2 = \frac{k_1}{\gamma_1}x_1 - k_2\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)x_2. \end{cases}$$

Initial conditions: $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ $\Rightarrow$

2x2 IVP: $\begin{cases} \dot{x}_1 = -\frac{k_1}{\gamma_1}x_1 + \frac{k_2}{\gamma_2}x_2, \\ \dot{x}_2 = \frac{k_1}{\gamma_1}x_1 - k_2\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)x_2. \end{cases}$ subject to $\begin{cases} x_1(0) = x_{10}, \\ x_2(0) = x_{20}. \end{cases}$

Systems can be written more compactly by using the matrix/vector notation.

An $n\times n$ matrix: A vector in $\mathbb{R}^n$:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\Rightarrow$ Definition: $$Ax = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}$$

$\Rightarrow$ If we denote $x' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$, $F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1, \\ x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2, \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n, \end{cases} \Rightarrow x' = Ax + F$$

In this notation,

Ex 1: $x' = Ax + F$, where $A = \begin{pmatrix} t & 2 & t^2 \\ e^t & e^{2t} & -3 \\ \cosh t & -t^3 & 1 \end{pmatrix}$, $F = \begin{pmatrix} \sin t \\ \cos t \\ -\sqrt{t} \end{pmatrix}$.

Ex 2: $x' = Ax$, where $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$.

Ex 3: $\begin{cases} \dot{x} = Ax, \\ x(0) = x_0. \end{cases}$ Here $A = \begin{pmatrix} -k_1/\gamma_1 & k_2/\gamma_2 \\ k_1/\gamma_1 & -k_2\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) \end{pmatrix}$ and $x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$.

Therefore, if $x' = Ax$, the system is homogeneous and, if $x' = Ax + F$, the system is nonhomogeneous.

Ex. Verify that $x_1 = e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^t \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ and $x_2 = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

solve $x' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} x$.

(a) $x_1 = \begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix} \Rightarrow x_1' = \begin{pmatrix} e^t + 4e^t + 4te^t \\ 3e^t - 4e^t - 4te^t \end{pmatrix}$

$= \begin{pmatrix} 5e^t + 4te^t \\ -e^t - 4te^t \end{pmatrix}$;

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix} = \begin{pmatrix} 2(e^t + 4te^t) + 3e^t - 4te^t \\ -(e^t + 4te^t) \end{pmatrix}$$

$= \begin{pmatrix} 5e^t + 4te^t \\ -(e^t + 4te^t) \end{pmatrix}$ – this is the same as x_1' above $\Rightarrow x_1' = Ax_1$

(b) $X_2 = \begin{pmatrix} e^t \\ -e^t \end{pmatrix} \Rightarrow X_2' = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$;

$$AX_2 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} 2e^t - e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t \end{pmatrix} -$$

the same as $X_2' \Rightarrow X_2' = AX_2$

(*) Superposition principle for systems: suppose that $X_1, \dots, X_k$ solve $X' = AX \Rightarrow c_1X_1 + c_2X_2 + \dots + c_kX_k$ solve $X' = AX$ for any constants $c_1, \dots, c_k$.

We will demonstrate this for 2×2 systems. Suppose that $X_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$ and $X_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$ solve $X' = AX$, where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then

(a) $$(c_1X_1 + c_2X_2)' = \left(c_1 \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} + c_2 \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}\right)' = \begin{pmatrix} c_1x_{11} + c_2x_{21} \\ c_1x_{12} + c_2x_{22} \end{pmatrix}'$$

$$= \begin{pmatrix} c_1x_{11}' + c_2x_{21}' \\ c_1x_{12}' + c_2x_{22}' \end{pmatrix} = c_1 \begin{pmatrix} x_{11}' \\ x_{12}' \end{pmatrix} + c_2 \begin{pmatrix} x_{21}' \\ x_{22}' \end{pmatrix} = c_1 \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}' + c_2 \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}'$$

$$= c_1X_1' + c_2X_2' = c_1AX_1 + c_2AX_2 = c_1 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$

$$+ c_2 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = c_1 \begin{pmatrix} a_{11}x_{11} + a_{12}x_{12} \\ a_{21}x_{11} + a_{22}x_{12} \end{pmatrix} + c_2 \begin{pmatrix} a_{11}x_{21} + a_{12}x_{22} \\ a_{21}x_{21} + a_{22}x_{22} \end{pmatrix}$$

$$= \begin{pmatrix} c_1(a_{11}x_{11} + a_{12}x_{12}) + c_2(a_{11}x_{21} + a_{12}x_{22}) \\ c_1(a_{21}x_{11} + a_{22}x_{12}) + c_2(a_{21}x_{21} + a_{22}x_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(c_1x_{11}+c_2x_{21}) + a_{12}(c_1x_{12}+c_2x_{22}) \\ a_{21}(c_1x_{11}+c_2x_{21}) + a_{22}(c_1x_{12}+c_2x_{22}) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_1x_{11}+c_2x_{21} \\ c_1x_{12}+c_2x_{22} \end{pmatrix}$$

$= A(c_1x_1+c_2x_2)$ – here we use the definition of matrix-vector product and properties of vector functions.

(*) Linear independence of vectors:

vectors $x_1, \dots, x_k$ are linearly independent if

$$c_1x_1+c_2x_2+\dots+c_kx_k=0 \Rightarrow c_1=c_2=\dots=c_k=0.$$

Otherwise the set of vectors is linearly dependent.

[in other words, $x_1, \dots, x_k$ if one of the vectors can be written as a linear combination of other vectors from the same set]

Ex: Are vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ linearly independent? Suppose that

$$c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{cases} c_1+2c_2=0 \\ 2c_1+3c_2=0 \end{cases} \Rightarrow \begin{cases} 2c_1+4c_2=0 \\ 2c_1+3c_2=0 \end{cases} \text{subtract} \Rightarrow c_2=0 \Rightarrow$$

$c_1=0 \Rightarrow$ the vectors are linearly independent.

Ex: Are

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}3\\2\\3\end{pmatrix} \text{ linearly independent?}$$

By inspection,

$$\begin{pmatrix}3\\2\\3\end{pmatrix} = 3\begin{pmatrix}1\\0\\1\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix}$$

– the three vectors are linearly dependent.

(*) n solutions $x_1, \dots, x_n$ of an $n\times n$ system are linearly independent if the Wronskian

$$W(x_1 x_2 \dots x_n) = \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & & x_{n2} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

Ex: Are $x_1 = e^t\begin{pmatrix}1\\3\end{pmatrix} + te^t\begin{pmatrix}4\\-4\end{pmatrix}$ and $x_2 = e^t\begin{pmatrix}1\\-1\end{pmatrix}$ linearly independent?

$$x_1 = \begin{pmatrix}e^t + 4te^t\\ 3e^t - 4te^t\end{pmatrix}, \quad x_2 = \begin{pmatrix}e^t\\ -e^t\end{pmatrix}$$

$$\Rightarrow W(x_1, x_2) = \begin{vmatrix} e^t + 4te^t & e^t \\ 3e^t - 4te^t & -e^t \end{vmatrix} = -e^t(e^t + 4te^t) - e^t(3e^t - 4te^t)$$

$= -e^{2t} - \cancel{4te^{2t}} - 3e^{2t} + \cancel{4te^{2t}} = -4e^{2t} \neq 0 \Rightarrow$ linearly independent.

(*) Given an $n \times n$ system $x' = Ax$, if n solutions

$$x_1, \dots, x_n$$

are linearly independent, then they form a fundamental set of solutions and the general solution of the system is $x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Ex: We showed above that $x_1 = e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^t \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ and $x_2 = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are 2 linearly independent solutions of the 2×2 system

$$x' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} x \Rightarrow$$

General solution is $x = c_1 \left(e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^t \begin{pmatrix} 4 \\ -4 \end{pmatrix} \right) + c_2 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Ex: Solve the initial value problem

$$\begin{cases} x' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} x \\ x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{cases}$$

Because $x(t) = c_1 \left(e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^t \begin{pmatrix} 4 \\ -4 \end{pmatrix} \right) + c_2 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix}1\\2\end{pmatrix} = x(0) = c_1\left(e^0\begin{pmatrix}1\\3\end{pmatrix} + 0e^0\begin{pmatrix}4\\-4\end{pmatrix}\right) + c_2e^0\begin{pmatrix}1\\-1\end{pmatrix} = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1+c_2\\3c_1-c_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \Rightarrow \begin{cases}c_1+c_2=1\\3c_1-c_2=2\end{cases} \text{ add: } 4c_1=3 \Rightarrow c_1=\frac{3}{4}$$

$\Rightarrow$ From the first equation $c_2 = 1-c_1 = 1-\frac{3}{4} = \frac{1}{4} \Rightarrow$

$$x(t) = \frac{3}{4}\left(e^t\begin{pmatrix}1\\3\end{pmatrix} + te^t\begin{pmatrix}4\\-4\end{pmatrix}\right) + \frac{1}{4}e^t\begin{pmatrix}1\\-1\end{pmatrix} = e^t\begin{pmatrix}1\\2\end{pmatrix} + 3te^t\begin{pmatrix}1\\-1\end{pmatrix}$$

(*) Linear homogeneous systems are equivalent to higher order homogeneous ODEs:

<u>Ex</u> $y'' + 3y' + 2y = 0 \Rightarrow$ Let $y_1 = y$, $y_2 = y' \Rightarrow y'' = y_2'$

$$\Downarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\Rightarrow \boxed{y_2' + 3y_2 + 2y_1 = 0} \qquad\qquad \boxed{y_1' = y_2}$$

$$\begin{cases}y_1' = y_2\\ y_2' = -2y_1 - 3y_2\end{cases} \quad - \text{ 2x2 system}$$

<u>Ex</u> $x' = \begin{pmatrix}2 & 1\\-1 & 0\end{pmatrix}x \Leftrightarrow \begin{cases}x_1' = 2x_1 + x_2\\ \underbrace{x_2' = -x_1}_{x_1 = -x_2'}\end{cases}$ Eliminate x_1 $\Leftrightarrow$

$$\begin{cases}x_1' = 2x_1 + x_2\\ x_2'' = -x_1'\end{cases} \Rightarrow x_2'' = -(2x_1 + x_2) \Rightarrow x_2'' = -(-2x_2' + x_2)$$

$\Rightarrow x_2'' - 2x_2' + x_2 = 0$ – can solve this second order ODE for $x_2 \Rightarrow x_1 = -x_2'$