Ex 1: Solve

$$\begin{cases} x' = 2x + 3y \\ y' = 2x + y \end{cases} \quad \text{(ODE)} \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases} \quad \text{(IC)}$$

Set $\overline{X} = \mathcal{L}\{x\}$, $\overline{Y} = \mathcal{L}\{y\}$ $\Rightarrow$ transform both ODEs:

$$\begin{cases} s\overline{X} - x_0 = 2\overline{X} + 3\overline{Y} \\ s\overline{Y} - y_0 = 2\overline{X} + Y \end{cases}$$ – system of algebraic equations.

$$\Rightarrow \begin{cases} (s-2)\overline{X} - 3\overline{Y} = x_0 \\ -2\overline{X} + (s-1)\overline{Y} = y_0 \end{cases} \Rightarrow \begin{cases} 2(s-2)\overline{X} - 6\overline{Y} = 2x_0 \\ -2(s-2)\overline{X} + (s-1)(s-2)\overline{Y} = (s-2)y_0 \end{cases}$$

add: $(s-1)(s-2)\overline{Y} - 6\overline{Y} = 2x_0 + (s-2)y_0$

$$(s^2 - 3s + 2 - 6)\overline{Y} = 2x_0 + (s-2)y_0$$

$$(s^2 - 3s - 4)\overline{Y} = 2x_0 + (s-2)y_0$$

$$(s-4)(s+1)\overline{Y} = 2x_0 + (s-2)y_0$$

$$\overline{Y} = \frac{2x_0 + (s-2)y_0}{(s-4)(s+1)} = \frac{A}{s-4} + \frac{B}{s+1} = \frac{A(s+1) + B(s-4)}{(s-4)(s+1)}$$

$A(s+1) + B(s-4) = 2x_0 + (s-2)y_0 \Rightarrow$

$s = -1$: $B(-1-4) = 2x_0 + (-1-2)y_0 \Rightarrow -5B = 2x_0 - 3y_0$

$\Rightarrow B = -\frac{2}{5}x_0 + \frac{3}{5}y_0$

$s = 4$: $5A = 2x_0 + (4-2)y_0 = 2x_0 + 2y_0 \Rightarrow A = \frac{2}{5}x_0 + \frac{2}{5}y_0$

$$\Rightarrow \quad \bar{Y} = \left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\frac{1}{s-4} + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\frac{1}{s+1}$$

$\Rightarrow$ Because $-2\bar{X} + (s-1)\bar{Y} = y_0$,

$$\bar{X} = \frac{1}{2}\left((s-1)\bar{Y} - y_0\right) = \frac{1}{2}\left(\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\frac{s-1}{s-4}\right.$$

$$\left. + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\frac{s-1}{s+1} - y_0\right) = \frac{1}{2}\left(\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\frac{s-4+3}{s-4}\right.$$

$$\left. + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\frac{s+1-2}{s+1} - y_0\right) = \frac{1}{2}\left(\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\left(1+\frac{3}{s-4}\right)\right.$$

$$+ \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\left(1-\frac{2}{s+1}\right) - y_0 = \frac{1}{2}\left(\cancel{\frac{2}{5}x_0}+\cancel{\frac{2}{5}y_0}\right.$$

$$\left. +3\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\frac{1}{s-4} \cancel{-\frac{2}{5}x_0}+\cancel{\frac{3}{5}y_0} - 2\left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\frac{1}{s+1} \cancel{-y_0}\right)$$

$$= \frac{3}{2}\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\frac{1}{s-4} - \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\frac{1}{s+1}$$

$$y = \mathcal{L}^{-1}\{\bar{Y}\} = \mathcal{L}^{-1}\left\{\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\frac{1}{s-4} + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\frac{1}{s+1}\right\}$$

$$= \left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)e^{4t} + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t}$$

$$x = \mathcal{L}^{-1}\{\bar{X}\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\frac{1}{s-4} - \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\frac{1}{s+1}\right\}$$

$$= \frac{3}{2}\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{3}{2}\left(\frac{2}{5}x_0+\frac{2}{5}y_0\right)e^{4t} - \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t}$$

$$\begin{cases} x(t) = \frac{3}{5}(x_0+y_0)e^{4t} - \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t} \\ y(t) = \frac{2}{5}\underbrace{(x_0+y_0)}_{5c_1}e^{4t} + \underbrace{\left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)}_{c_2}e^{-t} \end{cases}$$ – solution of (IVP)

$$\Rightarrow \begin{cases} x(t) = 3c_1e^{4t} - c_2e^{-t} \\ y(t) = 2c_1e^{4t} + c_2e^{-t} \end{cases}$$ – general solution of the system of ODEs

Note that the original (IVP) can be written as:

$$\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{cases}$$ $\Rightarrow$ solution in vector form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{3}{5}(x_0+y_0)e^{4t} - \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t} \\ \frac{2}{5}(x_0+y_0)e^{4t} + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{5}(x_0+y_0)e^{4t} \\ \frac{2}{5}(x_0+y_0)e^{4t} \end{pmatrix} + \begin{pmatrix} -\left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t} \\ \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t} \end{pmatrix}$$

$$= \frac{1}{5}(x_0+y_0)e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \left(-\frac{2}{5}x_0+\frac{3}{5}y_0\right)e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= c_1e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$ – general solution of the system in vector form.

Ex: Find the solution of the IVP:

$$\begin{cases} x' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} x \\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} x_1' = 3x_1 - 18x_2, \\ x_2' = 2x_1 - 9x_2, \end{cases} \quad \begin{cases} x_1(0) = 1, \\ x_2(0) = 0, \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{L}\{x_1'\} = 3\mathcal{L}\{x_1\} - 18\mathcal{L}\{x_2\} \\ \mathcal{L}\{x_2'\} = 2\mathcal{L}\{x_1\} - 9\mathcal{L}\{x_2\} \end{cases}$$

$$\Rightarrow \begin{cases} s\bar{X}_1 - 1 = 3\bar{X}_1 - 18\bar{X}_2 \\ s\bar{X}_2 = 2\bar{X}_1 - 9\bar{X}_2 \end{cases}$$

$$\Rightarrow \begin{cases} (s-3)\bar{X}_1 + 18\bar{X}_2 = 1 \\ 2\bar{X}_1 - (s+9)\bar{X}_2 = 0 \Rightarrow \bar{X}_1 = \frac{1}{2}(s+9)\bar{X}_2 \end{cases}$$

sub into 1st eqn $\Rightarrow$

$$(s-3)\cdot\frac{1}{2}(s+9)\bar{X}_2 + 18\bar{X}_2 = 1$$

$$(s^2+6s-27)\bar{X}_2 + 36\bar{X}_2 = 2$$

$$(s^2+6s-27+36)\bar{X}_2 = 2$$

$$(s^2+6s+9)\bar{X}_2 = 2 \Rightarrow \bar{X}_2 = \frac{2}{s^2+6s+9} = \frac{2}{(s+3)^2}$$

$$\Rightarrow \bar{X}_1 = \frac{1}{2}(s+9)\bar{X}_2 = \frac{1}{\cancel{2}}(s+9)\frac{\cancel{2}}{(s+3)^2} = \frac{s+9}{(s+3)^2}$$

$$\Rightarrow \quad x_1 = \mathcal{L}^{-1}\left\{\frac{s+9}{(s+3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3+6}{(s+3)^2}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{s+6}{s^2}\right\}$$

$$= e^{-3t}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right) = e^{-3t}(1+6t)$$

because $\mathcal{L}\{f(t)e^{at}\} = \mathcal{L}\{f(t)\}(s-a) \Leftrightarrow$

$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$.

$$x_2 = \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = 2e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 2e^{-3t}t$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{-3t}(1+6t) \\ 2e^{-3t}t \end{pmatrix} = e^{-3t}\begin{pmatrix} 1+6t \\ 2t \end{pmatrix}$$

$$= e^{-3t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 6t \\ 2t \end{pmatrix}\right) = e^{-3t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 6 \\ 2 \end{pmatrix}\right)$$

<u>Ex:</u> $\quad x' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- find the solution of the IVP by using the L.T.

If $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} x_1' = 6x_1 - x_2 \\ x_2' = 5x_1 + 4x_2 \end{cases} \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 1 \end{cases}$

$$\Rightarrow \begin{cases} \mathcal{L}\{x_1'\} = 6\mathcal{L}\{x_1\} - \mathcal{L}\{x_2\} \\ \mathcal{L}\{x_2'\} = 5\mathcal{L}\{x_1\} + 4\mathcal{L}\{x_2\} \end{cases} \Rightarrow \begin{cases} s\underline{\bar{X}}_1 = 6\underline{\bar{X}}_1 - \underline{\bar{X}}_2 \\ s\underline{\bar{X}}_2 - 1 = 5\underline{\bar{X}}_1 + 4\underline{\bar{X}}_2 \end{cases}$$

First equation: $\underline{\bar{X}}_2 = -(s-6)\underline{\bar{X}}_1 \Rightarrow$ sub this into the second equation: $-s(s-6)\underline{X}_1 - 1 = 5\underline{X}_1 - 4(s-6)\underline{X}_1$

$$\Rightarrow \left[-s(s-6) - 5 + 4(s-6)\right]\underline{X}_1 = 1$$

$$(-s^2+6s-5+4s-24)\underline{X}_1 = 1$$

$$(-s^2+10s-29)\underline{X}_1 = 1 \Rightarrow \underline{X}_1 = \frac{1}{-s^2+10s-29} = -\frac{1}{s^2-10s+29}$$

$$\Rightarrow \underline{X}_2 = -(s-6)\underline{X}_1 = \frac{s-6}{s^2-10s+29};$$

$$x_1 = \mathcal{L}^{-1}\left\{-\frac{1}{s^2-10s+29}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2-10s+29}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2-10s+25+4}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^2+4}\right\} = -\frac{1}{2}e^{5t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = -\frac{1}{2}e^{5t}\sin 2t.$$

$$x_2 = \mathcal{L}^{-1}\left\{\frac{s-6}{s^2-10s+29}\right\} = \mathcal{L}^{-1}\left\{\frac{s-6}{(s-5)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s-5-1}{(s-5)^2+4}\right\}$$

$$= e^{5t}\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+4}\right\} = e^{5t}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right)$$

$$= e^{5t}\left(\cos 2t - \frac{1}{2}\sin 2t\right)$$

$$\Rightarrow x = \begin{pmatrix} -\frac{1}{2}e^{5t}\sin 2t \\ e^{5t}\cos 2t - \frac{1}{2}e^{5t}\sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}e^{5t}\sin 2t \\ -\frac{1}{2}e^{5t}\sin 2t \end{pmatrix} + \begin{pmatrix} 0 \\ e^{5t}\cos 2t \end{pmatrix}$$

$$= -\frac{1}{2}e^{5t}\sin 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{5t}\cos 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$