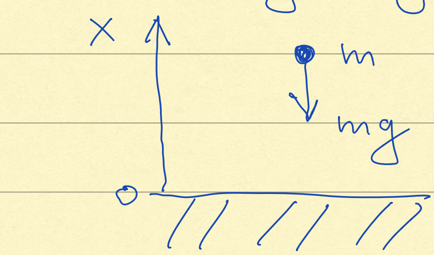
(1) Falling body:



Position of the ball at time t is

$$x(t)$$

Assume no air resistance. Then, at the time $t=0$,

$$x(0) = x_0 \text{ - initial height.}$$

Want to find $x(t)$: Use the 2^{nd} Newton's law, $ma = F \Rightarrow \cancel{m}a = -\cancel{m}g \Rightarrow a = -g$, where $g = 9.81 \frac{m}{s^2}$. Recall

(1) Position is $x(t)$, velocity is $v(t)$

$$\Rightarrow x'(t) = v(t)$$

(2) Velocity is $v(t)$, acceleration is $a(t)$

$$\Rightarrow v'(t) = a(t)$$

Combining (1) and (2):

$$x''(t) = (x'(t))' = (v(t))' = v'(t) = a(t)$$

$$\Rightarrow \quad x''(t) = -\underbrace{g}_{\text{constant.}}$$

- this is an equation for an unknown function $x(t)$. To solve, we need to integrate:

$$v(t) = x'(t) = \int x''(t)\,dt = -\int g\,dt = -gt + c_1$$

$$x(t) = \int x'(t)\,dt = \int v(t)\,dt = \int (c_1 - gt)\,dt$$

$$= \int c_1\,dt - g\int t\,dt = c_1 t - g\frac{t^2}{2} + c_2 \qquad (\#)$$

To determine c_1 and c_2, use the condition $x(0) = x_0$ and also, we will assume that the ball was released from rest, i.e., $v(0) = 0 \Leftrightarrow x'(0) = 0$: From $(\#)$, we know that

$$x_0 = x(0) = c_1 \cdot \cancel{0}^{\,0} - \frac{g}{\cancel{2}} \cdot \cancel{0^2}^{\,0} + c_2 = c_2 \Rightarrow c_2 = x_0$$

$$0 = v(0) = -g \cdot 0 + c_1 \Rightarrow c_1 = 0.$$

To summarize, we solved

Initial value problem (IVP)

$$\begin{cases} x''(t) = -g & \leftarrow \text{differential equation} \\ \left.\begin{array}{l} x(0) = x_0 \\ x'(0) = 0 \end{array}\right\} & \leftarrow \text{initial conditions} \end{cases}$$

and the solution ended up being

$$x(t) = x_0 - \frac{gt^2}{2}$$

(2) Radioactive decay: Initially have a radioactive sample of mass m_0 with half-life $T_{1/2}$. What is the mass of the sample at time t?

Suppose the mass of the sample at time t is $m(t)$.

Facts: (1) $m(t)$ is a decreasing function $\Leftrightarrow$ $m'(t) < 0$

(2) $m(t+T_{1/2}) = \frac{1}{2} m(t)$, in particular, $m(T_{1/2}) = \frac{1}{2} m(0) = \frac{1}{2} m_0$

(3) Larger samples decay more quickly. We will make a simple assumption:

Rate of decay $= m'(t)$ is proportional to the mass at time $t \Rightarrow m'(t) = -k m(t)$

where "–" in front of $k>0$
guarantees decay.

(IVP)
$\Rightarrow$
$$\begin{cases} m'(t) = -km(t) & \text{– differential equation} \\ m(0) = m_0 & \leftarrow \text{initial condition} \end{cases}$$

Try to solve:

$m' = -km \iff m(t) = \int m'(t)\,dt = -k\int m(t)\,dt$

$\iff m(t) = -k\int m(t)\,dt$ – the integral cannot be evaluated explicitely
↑ unknown!

Try guessing the solution: try $m(t) = e^{-kt}\ldots$

$\Rightarrow m'(t) = e^{-kt}(-k) = -km(t)$ (*)

What to do with an unknown constant?

Can it be true that $m(t) = e^{-kt} + C$?

then $m'(t) = -ke^{-kt} = -k(m(t) - C)$ – wrong equation! Try $m(t) = Ce^{-kt} \Rightarrow$

$m'(t) = Ce^{-kt}(-k) = -k(Ce^{-kt}) = -km(t)$

– this $m(t)$ satisfies the correct equation (*)!

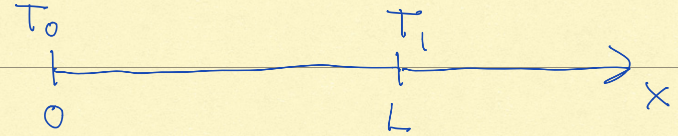
Now, solve the initial value problem:

$$m_0 = m(0) = Ce^{-k\cdot 0} = C \quad \Rightarrow$$

$$m(t) = m_0 e^{-kt}$$

Note that k must be related ~~to~~ to half-life $T_{1/2}$.

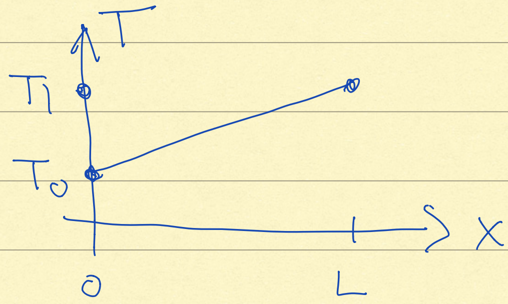
(3) Suppose that we have a bar one end of which is maintained at temperature T_0 while the other end is at temperature T_1. If initially the temperature across the bar is distributed in some prescribed way, what would the temperature be at every point in the bar at any time t?



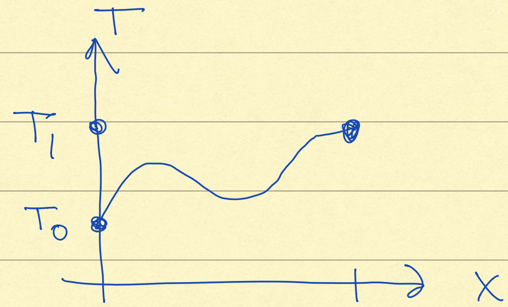
Want to know $T(x,t)$ - function of two variables.

$$\begin{cases} T(x,0) = T_{in}(x) \\ T(0) = T_0 \\ T(L) = T_1 \end{cases}$$

initial & boundary conditions.



- this temperature profile will not change in time.



- this temperature profile is not stationary

Can conclude that $T_t = kT_{xx}$

differential equation