A differential equation is any equation that contains an unknown function and its derivatives.

Examples of differential equations:

(a) $y' + 2y - x = 0$ (b) $y''y' + y^2 x = \sin x$

(c) $y^{(V)} + y''' - y^2 + y = e^x$ (d) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Immediately, we can classify these equations according to the type of derivatives that are involved:

(1) Ordinary derivatives $\Rightarrow$ ordinary differential equation (ODE)

(2) Partial derivatives $\Rightarrow$ partial differential equation (PDE)

Hence, (a) - (c) above are ODEs and (d) is a PDE

In this course we are only concerned with ODEs.

Next, we can classify differential equations according to the order of the highest derivative that appears in it $\Rightarrow$ (a) is the first order ODE, (b) is the second order ODE, (c) is the ODE of order 5 and (d) is the second order PDE. One general consequence of the order of a given ODE is that it tells us how many indefinite integrals one needs to take to go from the highest derivative of an unknown function to the function itself. E.g., to solve the equation (a), we will need to integrate once and this will introduce an unknown constant into the solution $\Rightarrow$ the equation (a) should have infinitely many solutions obtained by giving different values to one unknown constant. Similarly, to solve (b), we will have to integrate twice and there will be two unknown constants. To solve (c) we will have to integrate five times so that the solution should depend on five unknown constants. To summarize, the order of an ODE dictates the number of unknown constants that should appear in the solution.

Example: Classify and solve the equation

$$y''' = \sin x$$

- this is a third order ODE $\Rightarrow$

$$y'' = \int y''' dx = \int \sin x\, dx = -\cos x + C_1$$

$y' = \int y'' dx = \int (-\cos x + c_1)\, dx = -\sin x + c_1 x + c_2$

$y = \int y' dx = \int (-\sin x + c_1 x + c_2)\, dx = \cos x + \frac{c_1 x^2}{2} + c_2 x + c_3$

– there are 3 unknown constants.

we will call this a three-parameter family of solutions of our ODE.

Thus, in general, an n-th order ODE posesses an n-parameter family of solutions.

Example: Verify that $y = c_1 \cos x + c_2 \sin x$ is a two-parameter family of solution of the second order ODE $y'' + y = 0$.

To verify, we need to substitute y into the equation. First, compute

$$y'(x) = -c_1 \sin x + c_2 \cos x$$

$$y''(x) = -c_1 \cos x - c_2 \sin x$$

$\Rightarrow y''(x) + y(x) = \cancel{-c_1 \cos x} \cancel{- c_2 \sin x} + \cancel{c_1 \cos x} + \cancel{c_2 \sin x} = 0$

$\Rightarrow y = c_1 \cos x + c_2 \sin x$ solves $y'' + y = 0$ for any combination of c_1, c_2

$\Rightarrow$ $y = c_1 \cos x + c_2 \sin x$ is a two-parameter family of solutions of $y'' + y = 0$

Clearly, if we choose $c_1 = 1$, $c_2 = 0$, then $y = \cos x$ is a solution of $y'' + y = 0$ that does not involve any unknown constants. This is a <u>particular</u> solution of $y'' + y = 0$. Other examples of particular solutions of $y'' + y = 0$ are $y = \sin x$, $y = \sin x + \cos x$, $y = 2\sin x - \cos x$, etc.

Example: Find a particular solution of $y'' + y = 0$ that satisfies the conditions $y(0) = 1$, $y'(0) = -1$

To solve, use the two parameter family of solutions we found above:

$$y(x) = c_1 \cos x + c_2 \sin x \Rightarrow y'(x) = -c_1 \sin x + c_2 \cos x$$

$$1 = y(0) = c_1 \cancelto{1}{\cos 0} + c_2 \cancelto{0}{\sin 0} = c_1$$

$$-1 = y'(0) = -c_1 \cancelto{0}{\sin 0} + c_2 \cancelto{1}{\cos 0} = c_2$$

$$\Rightarrow c_1 = 1,\ c_2 = -1 \Rightarrow y(x) = \cos x - \sin x$$

In general, an n-th order ODE can be written in a form

$$F(x, y, y', \dots, y^{(n-1)}, y^{(n)}) = 0$$

where F is some function of $n+2$ variables, for example,

$$x^2 y'' + 2xy - 1 = 0 \quad \text{or} \quad e^x y' + y^2 = 0$$

These equation can be solved for their highest derivative to obtain

$$y'' = \frac{1-2xy}{x^2} \quad \text{or} \quad y' = -e^{-x} y^2$$

we will then say that these equations are written in their <u>normal form</u>:

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

Classification by linearity: Any equation of the form

any functions of x

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = f(x)$$

is a <u>linear</u> equation. All other equations are called <u>nonlinear</u>. For example:

$x^2y''+xy'+y=\sin x$ – second order, linear ODE.

$xy^{(5)}+x^2y''-xy'=0$ – fifth order, linear ODE.

$xy''+\sin x\, y^2=y$ – second order, nonlinear ODE.

$yy'+y'''=y^2$ – third order, nonlinear ODE.

More examples: (*) Verify that $y=x^3$ solves $x^2y''-xy'-3y=0$ and classify the ODE.

This is a second order, linear ODE. To check if x^3 is a solution, substitute into the ODE:

$$y'=3x^2,\quad y''=6x \Rightarrow$$

$$x^2y''-xy'-3y=x^2\cdot 6x-x\cdot 3x^2-3x^3=0. \checkmark$$

(*) Verify that $x^2+y^2=1$ is a solution of the ODE $yy'+x=0$. This is a first order nonlinear ODE. We need to substitute the given function into the ODE, however, the function $y(x)$ is given by $x^2+y^2=1$ implicitly. Need to take the derivative implicitly:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) \Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Chain Rule $\Rightarrow$ $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$. Substitute this back into the equation:

$$yy' + x = y\left(-\frac{x}{y}\right) + x = -x + x = 0$$

$\therefore$ $x^2 + y^2 = 1$ is an implicit solution of

$$yy' + x = 0.$$