Consider an n-th order differential equation in the normal form:

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{(n-1)}y}{dx^{n-1}}\right)$$

In the previous lecture we learned that a solution to this problem will generally depend on n generic constants $c_1, \dots, c_n$ so that we need to specify n additional conditions to find a particular solution to our ODE. One way to do this is to specify the values of y and its derivatives at some "initial" value of x, for example:

$$\left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}\right\} - n \text{ conditions}$$

$\Rightarrow$ Want to solve the initial value problem (IVP)

$$\begin{cases} \dfrac{d^n y}{dx^n} = f\left(x, y, \dfrac{dy}{dx}, \dots, \dfrac{d^{(n-1)}y}{dx^{n-1}}\right) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

Example from physics: Recall that is the first lecture we considered a ball falling to the ground from rest. If the altitude of the ball at time t is $x(t)$, then the velocity of the ball is $x'(t)$ and its acceleration is $x''(t)$. Then, from the 2^{nd} Newton's Law:

$$x''(t) = -g \quad - \ 2^{nd} \text{ order ODE}$$

while $x(0) = x_0$ - initial altitude

$x'(0) = 0$ - initial velocity $\Rightarrow$

$x(t)$ can be found by solving the IVP

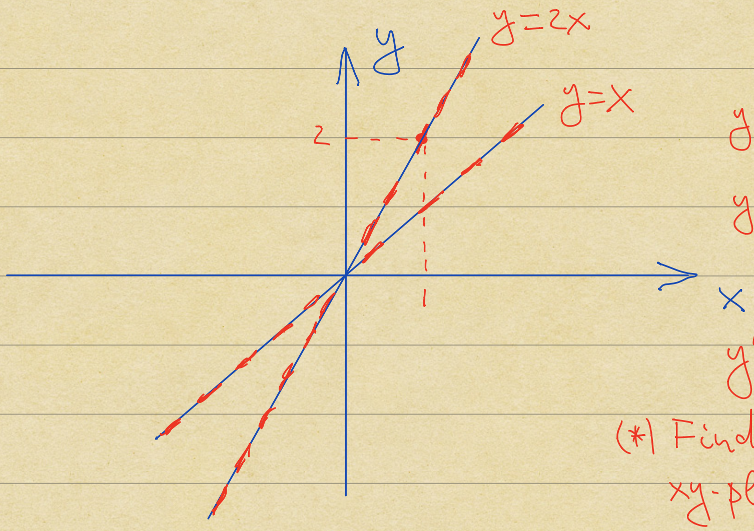
$$\begin{cases} x'' = -g \\ x(0) = x_0 \\ x'(0) = 0 \end{cases} \quad - \text{ second order IVP}$$

Geometric example: Suppose that we know that the slope of a curve when it passes through any point (x,y) on the xy-plane is equal to y/x. If we want to reconstruct such a curve from a starting point (x_0, y_0), then we need to solve the IVP

$$\begin{cases} y' = y/x \\ y(x_0) = y_0 \end{cases} \quad - \text{ first order IVP}$$

Let $(x_0, y_0) = (1,2) \Rightarrow$ solve

$$\begin{cases} y' = y/x \\ y(1) = 2 \end{cases} \Rightarrow y = 2x \text{ is the solution}$$



$y'(1,2) = \frac{2}{1} = 2$

$y'(1,1) = \frac{1}{1} = 1$

$y'(2,2) = \frac{2}{2} = 1$

(*) Find all points of xy-plane where the slope = 1:

$y/x = 1 \Rightarrow y = x$

(*) Find all points of xy-plane where the slope = 2:

$y/x = 2 \Rightarrow y = 2x$

Observation: $y = kx$ is a solution for all k:

Test: $y' = k = \frac{kx}{x} = \frac{y}{x}$. ✓

Example: Verify that $y'' - 4y = 0$ has a two-parameter family of solutions $y = c_1 e^{-2x} + c_2 e^{2x}$.

Solve the initial value problems:

$$(\mathrm{I}) \begin{cases} y'' - 4y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \quad \text{and} \quad (\mathrm{II}) \begin{cases} y'' - 4y = 0 \\ y(1) = 0 \\ y'(1) = e^2 \end{cases}$$

$$y' = c_1 e^{-2x}(-2) + c_2 e^{2x} \cdot 2 = 2\left(-c_1 e^{-2x} + c_2 e^{2x}\right)$$

$$y'' = c_1 e^{-2x}(-2)^2 + c_2 e^{2x} \cdot 2^2 = 4\left(c_1 e^{-2x} + c_2 e^{2x}\right)$$

$$y'' - 4y = 4\left(\cancel{c_1 e^{-2x}} + \cancel{c_2 e^{2x}}\right) - 4\left(\cancel{c_1 e^{-2x}} + \cancel{c_2 e^{2x}}\right) = 0$$

– y is is a solution for an arbitrary choice of c_1 and c_2

Solve (I): $\quad 1 = y(0) = c_1 e^{-2\cdot 0} + c_2 e^{2\cdot 0} = c_1 + c_2$

$$0 = y'(0) = 2\left(-c_1 e^{-2\cdot 0} + c_2 e^{2\cdot 0}\right) = 2(-c_1 + c_2)$$

$$\Rightarrow \begin{cases} c_2 - c_1 = 0 \\ c_2 + c_1 = 1 \end{cases} \Rightarrow 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2} \Rightarrow c_1 = c_2 = \frac{1}{2}$$

$\Rightarrow$ solution to (I) is $y(x) = \frac{1}{2} e^{-2x} + \frac{1}{2} e^{2x}$

Solve (II): $\quad 0 = y(1) = c_1 e^{-2\cdot 1} + c_2 e^{2\cdot 1} = c_1 e^{-2} + c_2 e^{2}$

$$e^2 = y'(1) = 2\left(-c_1 e^{-2\cdot 1} + c_2 e^{2\cdot 1}\right) = 2\left(-c_1 e^{-2} + c_2 e^{2}\right)$$

$$\Rightarrow \begin{cases} c_1 e^{-2} + c_2 e^{2} = 0 \\ -c_1 e^{-2} + c_2 e^{2} = \frac{1}{2} e^{2} \end{cases} \Rightarrow 2c_2 \cancel{e^{2}} = \frac{1}{2} \cancel{e^{2}}$$

$\Rightarrow c_2 = \frac{1}{4} \qquad c_1 e^{-2} + c_2 e^{2} = 0 \Rightarrow$

$$c_1 = -c_2 e^{2} / e^{-2} = -c_2 e^{4} = -\frac{1}{4} e^{4}$$

$$\Rightarrow y(x) = -\frac{1}{4} e^{4} e^{-2x} + \frac{1}{4} e^{2x}$$

Example: Verify that $y' + 3x^2y^2 = 0$ has a one-parameter family of solutions $y = (x^3+c)^{-1}$ and find solutions of IVP:

$$(\mathrm{I}) \begin{cases} y' + 3x^2y^2 = 0 \\ y(0) = 1 \end{cases} \quad \text{and} \quad (\mathrm{II}) \begin{cases} y' + 3x^2y^2 = 0 \\ y(1) = 0 \end{cases}$$

what are the intervals of definition for each of these problems?

$$y(x) = (x^3+c)^{-1} \implies y' = -(x^3+c)^{-2} \cdot 3x^2$$

Substitute these into the ODE

$$y' + 3x^2y^2 = -3x^2(x^3+c)^{-2} + 3x^2\left((x^3+c)^{-1}\right)^2$$

$$= -3x^2\cancel{(x^3+c)^{-2}} + \cancel{3x^2(x^3+c)^{-2}} = 0$$

Hence $y(x) = (x^3+c)^{-1}$ is a one-parameter family of solutions of $y' + 3x^2y^2 = 0$

Solve (I): $1 = y(0) = (0^3+c)^{-1} \implies c = 1 \implies$

solution of (I) is $y(x) = (x^3+1)^{-1}$

Solve (II): $0 = y(1) = (1^3+c)^{-1} \implies 0 = \dfrac{1}{c+1}$ – cannot solve for c! $\implies$ cannot find a solution from 1-parameter family.

Note that $y=0$ is a solution of (II) but it is not of the form $y=(x^3+C)^{-1}$.

Example: Show that $y\equiv 0$, $y=x^3$, and

$y=\begin{cases} x^3, & x\geq 0 \\ -x^3, & x<0 \end{cases}$ all solve the IVP

$$\begin{cases} y'=3xy^{1/3} \\ y(0)=0 \end{cases}$$

(1)

$y\equiv 0 \Rightarrow y'=0 \Rightarrow \begin{cases} y'=0 \\ 3xy^{1/3}=0 \end{cases}$ - equation is satisfied along with the initial condition.

(2) $y=x^3 \Rightarrow y'=3x^2$

$3xy^{1/3}=3x(x^3)^{1/3}=3x\cdot x=3x^2$

$\Rightarrow y'=3xy^{1/3}$ - equation is satisfied

also, $y(0)=0^3=0 \Rightarrow y=x^3$ solves (IVP)

(3) If $y=-x^3 \Rightarrow y'=-3x^2$ and

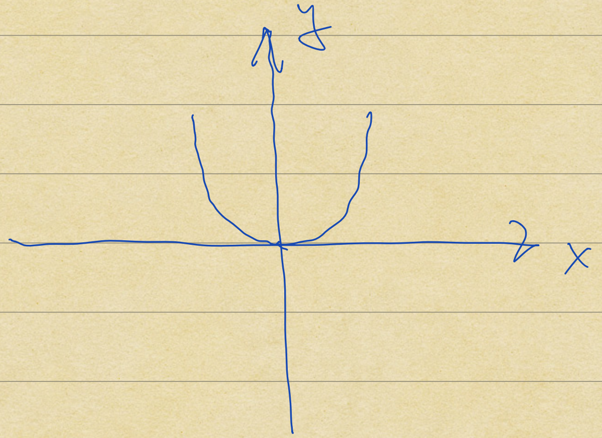
$3xy^{1/3}=3x(-x^3)^{1/3}=-3x(x^3)^{1/3}=-3x^2$

$\Rightarrow y=-x^3$ solves $y'=3xy^{1/3}$

also $-(0)^3=0$ - the initial condition

is satisfied $\Rightarrow y=-x^3$ also solves

IVP.



$\therefore$ IVP has at least 4 solutions

Example: Show that the IVP

$$\begin{cases} y' = (x+y)^{1/2} \\ y(0) = -1 \end{cases}$$

has no solutions.

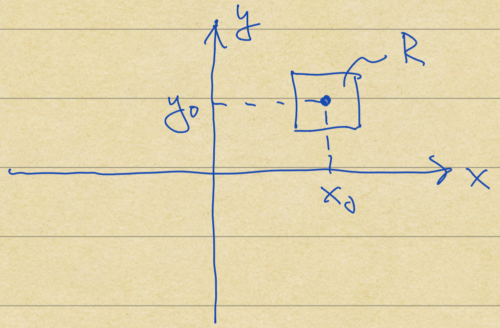
at the initial point $(0,-1)$: $y' = (0+(-1))^{1/2}$ - undefined

- no solution.

Want to know whether a solution of an initial value problem

$$\text{(IVP)} \qquad \begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

exists and is unique.



Suppose that there exists a rectangle R containing (x_0, y_0) and such that $f(x,y)$ and $f_y(x,y)$ are both continuous on R. Then (IVP) has the unique solution.

Example: Does $\begin{cases} y' + 3x^2y^2 = 0 \\ y(0) = 1 \end{cases}$ have a unique solution?

Write the equation in the normal form:

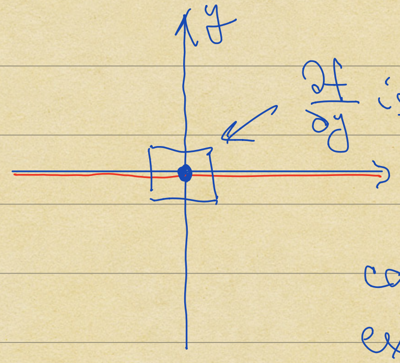
$$y' = -3x^2y^2 \implies \left.\begin{aligned} f(x,y) &= -3x^2y^2 \\ \frac{\partial f}{\partial y} &= -6xy \end{aligned}\right\} \text{ - both of these are continuous everywhere.}$$

$\Rightarrow$ IVP has a unique solution.

Example: Does the solution of $\begin{cases} y' = 3xy^{1/3} \\ y(0) = 0 \end{cases}$

exist? Is it unique?

In this case, $f(x,y) = 3xy^{1/3}$ – continuous for all (x,y). $\frac{\partial f}{\partial y} = 3 \cdot \frac{1}{3} x y^{1/3 - 1} = x y^{-2/3}$ - continuous as long as $y \neq 0$

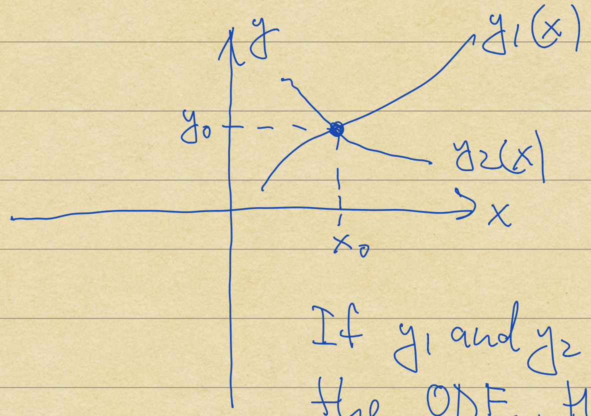


$\frac{\partial f}{\partial y}$ is discontinuous on any rectangle containing $(0,0)$ $\Rightarrow$ cannot guarantee existence and uniqueness of a solution to IVP.

Fact: Suppose that

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution.



If y_1 and y_2 both satisfy the ODE, this situation cannot happen.

$\Rightarrow$ If any initial value problem associated with an

ODE has a unique solution, then any two solution curves cannot intersect.