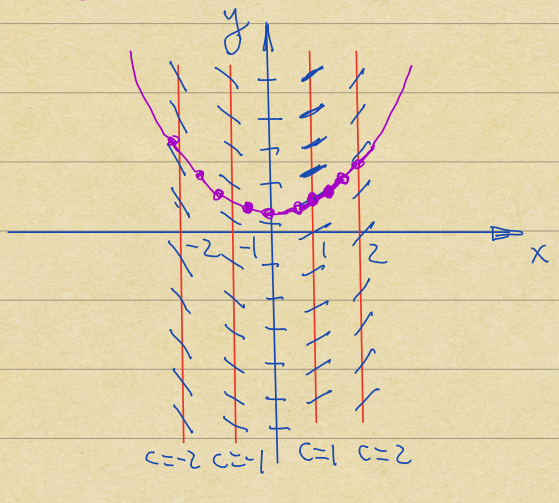
Given the ODE:

$$y' = f(x,y),$$

note, that the ODE specifies the slope of a tangent line to a curve that passes through a point (x,y). We can then plot a segment of a tangent line at each (x,y) where the function f is defined to give us the <u>direction field</u> associated with $y' = f(x,y)$; We can then use the direction field to graphically solve initial value problems.

Example: Sketch the direction field of $y' = x$ and use it to draw the solution curve for the IVP

$y' = x \Rightarrow y = \int x\,dx = \frac{x^2}{2} + C$; $1 = y(1) = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$
$\Rightarrow y = \frac{x^2}{2} + \frac{1}{2}$. $\Leftarrow$ $\begin{cases} y' = x \\ y(1) = 1 \end{cases}$



Find all points where

(a) $y' = 1$: since $y' = x$

$\Rightarrow 1 = y' = x \Rightarrow x = 1$

– vertical line through $x = 1$.

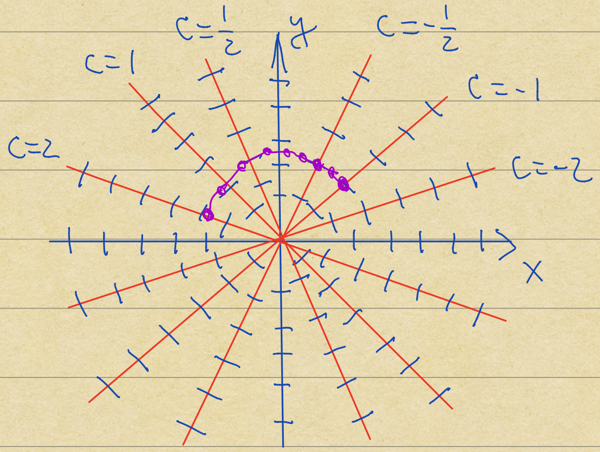
(b) $y' = -1$: $-1 = y' = x \Rightarrow x = -1$

(c) Find all points where $y' = c$ - constant $\Rightarrow$ looking at all points where $x = c$ - vertical lines.

Example: Sketch the direction field of $y' = -\frac{x}{y}$ and use it to draw the solution curve for the IVP

$$\begin{cases} y' = -\frac{x}{y} \\ y(1) = 1 \end{cases}$$

Check: $2x + 2yy' = 0$
$\Rightarrow y' = -x/y$ ✓
$\Uparrow$
solution seems to be $x^2 + y^2 = 2$
$\Uparrow$



Find all points where $y' = c \Rightarrow -\frac{x}{y} = c$

$\Rightarrow y = -\frac{x}{c} = -\frac{1}{c}x$

$\Rightarrow$ for each c, the set of points is a line through the origin w/slope $-\frac{1}{c}$.

Note: no particular direction at the origin!

$c = -1$, $y = x$; $c = 1 \Rightarrow y = -x$

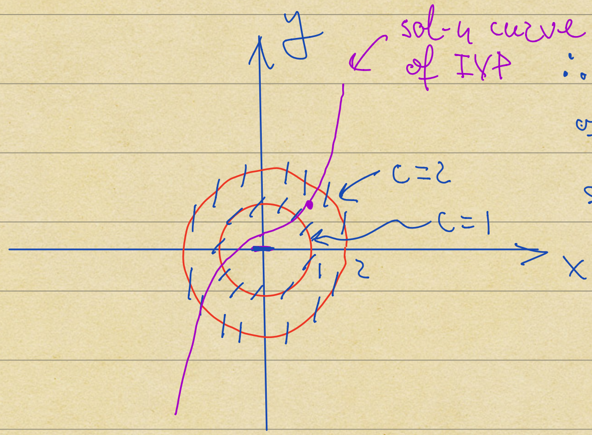
$c = -\frac{1}{2}$, $y = 2x$; $c = 2 \Rightarrow y = -\frac{1}{2}x$

$c = -2$, $y = \frac{1}{2}x$; $c = \frac{1}{2} \Rightarrow y = -2x$

Example: Sketch the direction field of $y' = x^2 + y^2$ and use it to draw the solution curve for the IVP

$$\begin{cases} y' = x^2 + y^2 \\ y(1) = 1 \end{cases}$$

Find all points where $y' = c^2$: $x^2 + y^2 = c^2$ – circle of radius c.



∴ slope ↑ on circles of larger radius

slopes on circles of radii c are $= c^2$.

Autonomous equations.

<u>Def:</u> A first order ODE is autonomous if

$y' = f(y)$ – f is independent of x.

Examples of autonomous equations:

(a) $y' = y$, (b) $y' = y(1-y)$, (c) $y' = \sin y$

On the other hand, $y' = x + y$ is not autonomous.

Suppose $y' = f(y)$ while f and $f'(y)$ are continuous functions of y. Then recall that

$$\begin{cases} y' = f(y) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution for any (x_0, y_0) $\Rightarrow$ any two solution curves of $y' = f(y)$ do not intersect.

Note: (1) if $f(y) > 0$, then $y' > 0$ $\Rightarrow$ a solution curve passing through (x,y) is $\uparrow$

(2) if $f(y) < 0$, then $y' < 0$ $\Rightarrow$ a solution curve passing through (x,y) is $\downarrow$

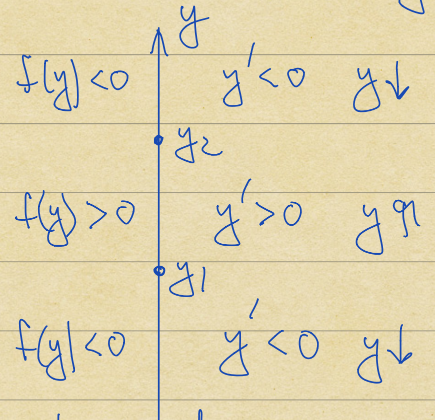
(3) Also, suppose that $f(y_s) = 0$ $\Rightarrow$ set $y = y_s$ – a constant function, so that $y' = 0$ $\Rightarrow$

$$y' = 0 = f(y_s) = f(y)$$

$\Rightarrow$ $y = y_s$ is a constant solution of $y' = f(y)$
– call these equilibrium solutions of $y' = f(y)$.

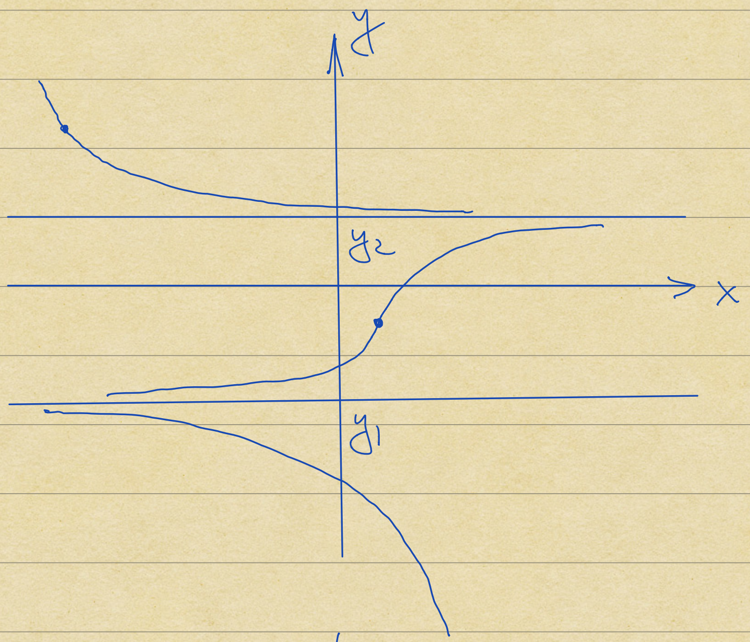
Sketch the diagram:

Suppose $f(y_1) = f(y_2) = 0$
$f > 0$ if $y \in (y_1, y_2)$
$f < 0$ otherwise



– phase diagram

y_1, y_2 are constant equilibrium solutions.



If we start solving at y slightly greater than $y_2 \Rightarrow$ as $x \to \infty$ $y \to y_2$

If we start solving at y slightly less than $y_2 \Rightarrow$ as $x \to \infty$ $y \to y_2$

If we start solving at y slightly greater than $y_1 \Rightarrow$ as $x \to \infty$ y goes away from y_1.

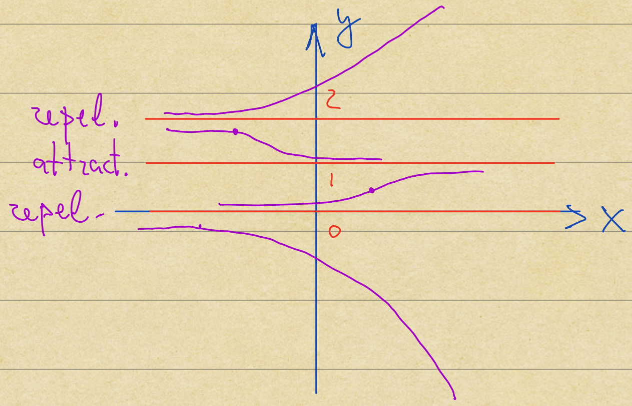
If we start solving at y slightly less than $y_2 \Rightarrow$ as $x \to \infty$ y goes away from y_1.

We call y_2 an <u>attractor</u>, y_1 a <u>repeller</u>.

Example: $y' = y(y-1)(y-2)$ – autonomous ODE. (r.h.s. is independent of x).

$f(y) = 0 \Leftrightarrow y(y-1)(y-2) = 0$

$\Rightarrow$ $y=0$, $y=1$, $y=2$ – are equilibrium solutions of the ODE.



Example: $y' = \sin y$ – autonomous ODE

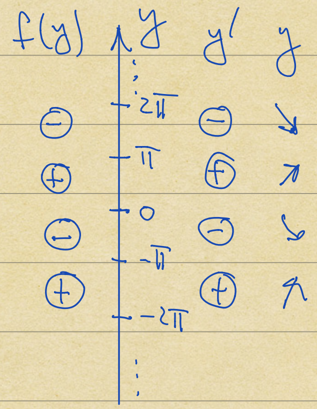
(1) Find equilibrium solutions:

$$\sin y = 0$$

$\Rightarrow$ $y = \pi n$, n is an integer

$\Rightarrow$ ∞ many equilibrium solutions.

(2)



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