<u>Def:</u> A first order ODE is separable if it is in the form

$$y' = f(x)g(y)$$

How to solve? (1) Look for solutions of $g(y)=0$.
Suppose that y_0 is such a solution, that is, $g(y_0)=0$.
Then consider a constant function $y=y_0$.
We know, $y' = y_0' = 0$ $\Rightarrow$ substitute this into the ODE:

$y'=0$ and $f(x)g(y) = f(x)g(y_0) = 0$ for any x.

$\Rightarrow$ $y' = f(x)g(y)$ $\Rightarrow$ $y=y_0$ is a solution

Call these equilibrium solution.

(2) Suppose that $y(x)$ is such that $g(y(x)) \neq 0$ for any x $\Rightarrow$ write the ODE in the form:

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{dy}{g(y)} = f(x)\,dx$$

$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$ $\Rightarrow$ implicit solution of the ODE.

Example: Solve $y' = \underbrace{y}_{g(y)}\underbrace{\sin x}_{f(x)}$ – 1st order separable ODE

(1) Solve $g(y)=0 \iff y=0$ – equilibrium solution of the ODE

(2) Separate variables:

$$\frac{dy}{dx}=y\sin x \Rightarrow \frac{dy}{y}=\sin x\,dx \Rightarrow \int\frac{dy}{y}=\int\sin x\,dx$$

$\Rightarrow \ln|y|=-\cos x+C$ – one-parameter family of solutions.

Solve IVP: $\begin{cases} y'=y\sin x \\ y(\frac{\pi}{2})=1 \end{cases}$ $\quad \underbrace{\ln|1|}_{0}=-\underbrace{\cos\frac{\pi}{2}}_{0}+C \Rightarrow C=0$

$\Rightarrow$ solution of IVP: $\ln|y|=-\cos x$ – implicit solution. Find the explicit solution:

$$e^{\ln|y|}=e^{-\cos x} \Rightarrow$$

$$|y|=e^{-\cos x} \Rightarrow y=\begin{cases} e^{-\cos x} \\ -e^{-\cos x} \end{cases}$$

Check the initial condition:

$$y(\tfrac{\pi}{2})=\begin{cases} e^{-\cos\pi/2} \\ -e^{-\cos\pi/2} \end{cases}=\begin{cases} 1 \\ -1 \text{ – not good} \end{cases}$$

– discard the second solution $\Rightarrow$

Solution of the IVP: $y=e^{-\cos x}$

Solve the IVP: $\begin{cases} y'=y\sin x \\ y(\pi)=-e \end{cases}$

$\ln|-e| = -\cos\pi + C \Rightarrow \ln e = -(-1) + C$

$1 = 1 + C \Rightarrow C = 0 \Rightarrow$ the same implicit solution applies: $\ln|y| = -\cos x \Rightarrow |y| = e^{-\cos x} \Rightarrow$

$y = \begin{cases} e^{-\cos x} \\ -e^{-\cos x} \end{cases}$. Check the initial condition:

$$y(\pi) = \begin{cases} e^{-\cos\pi} \\ -e^{-\cos\pi} \end{cases} = \begin{cases} e^{-(-1)} \\ -e^{-(-1)} \end{cases} = \begin{cases} e \\ -e \end{cases}$$

only the second solution satisfies the IVP

$$\Rightarrow y(x) = -e^{-\cos x}$$

Solve the IVP: $\begin{cases} y' = y\sin x \\ y(1) = 0 \end{cases}$

We know, that $\ln|y| = -\cos x + C \Leftarrow$ substitute the initial condition:

$$\underbrace{\ln|0|}_{\text{undefined}} = -\cos 1 + C$$

Does this mean that the IVP has no solutions?

Recall, if $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ and $f, \frac{\partial f}{\partial y}$ are continuous near $(x_0, y_0) \Rightarrow$ the IVP has a unique solution

Consider the IVP: $\begin{cases} y' = y\sin x \\ y(1) = 0 \end{cases}$

In this case, $f(x,y) = y\sin x$, $\frac{\partial f}{\partial y} = \sin x$ – continuous for all (x,y) $\Rightarrow$ the IVP has a unique solution.

As we just saw, this solution cannot be found from the one-parameter family of solutions. In this case, we use the equilibrium solution $y=0$.

Lastly, try to find a one-parameter family of explicit solutions:

$$\ln|y| = -\cos x + \underset{\uparrow}{C} \quad \text{arbitrary constant.}$$

$$\Rightarrow e^{\ln|y|} = e^{-\cos x + C}$$

$$\Rightarrow |y| = \underbrace{e^{C}}_{\text{positive constant } D} e^{-\cos x}$$

$$\Rightarrow |y| = De^{-\cos x} \Rightarrow y = \begin{cases} \underset{\text{arbitrary positive}}{D}e^{-\cos x} \\ \underset{\text{arbitrary negative}}{-D}e^{-\cos x} \end{cases}$$

$$\therefore\ y = \underset{\text{arbitrary constant.}}{E}e^{-\cos x}$$

Note: If $E=0 \Rightarrow y=0$ – that is, the equilibrium solution is a part of the one-parameter family of explicit solutions.

Example: Solve $xy' = y^2+1$. Is this separable?

$$\Rightarrow x\frac{dy}{dx} = y^2+1 \Rightarrow \frac{dy}{y^2+1} = \frac{dx}{x}$$

– separable

$$\Rightarrow \int \frac{dy}{y^2+1} = \int \frac{dx}{x} \Rightarrow \tan^{-1} y = \ln|x| + C$$ –

one parameter family of implicit solution; since $y^2+1 \neq 0 \Rightarrow$ no equilibrium solutions.

Explicit solution:

$$y = \tan(\ln|x| + C)$$

Example: Solve $y' = xy + y + x + 1$: Manipulate the right hand side:

$$y' = y(x+1) + (x+1) = (y+1)(x+1)$$

– separable ODE;

(1) Find equilibrium solutions: $y+1 = 0 \Rightarrow \boxed{y=-1}$

(2) $\frac{dy}{dx} = (y+1)(x+1) \Rightarrow \frac{dy}{y+1} = (x+1)dx$

$$\Rightarrow \int \frac{dy}{y+1} = \int (x+1)\,dx = \int x\,dx + \int dx = \frac{x^2}{2} + x + C$$

$u = y+1 \;||\; du = dy$

$$\int \frac{du}{u} = \ln|u| = \ln|y+1| \Rightarrow \boxed{\ln|y+1| = \frac{x^2}{2} + x + C}$$

Example: Solve $\frac{dy}{dx} = e^{3x+2y} = e^{3x}e^{2y}$ – separable

(1) $e^{2y} \neq 0$ – no equilibrium solutions

(2) $e^{3x}dx = e^{-2y}dy \Rightarrow \int e^{3x}dx = \int e^{-2y}dy$

$\Rightarrow \frac{1}{3}e^{3x} = -\frac{1}{2}e^{2y} + c$ – implit solution of the ODE

Example: $y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2 = \frac{(y+1)^2}{x^2}$

$x^2 \ln x \, dx = \frac{(y+1)^2}{y} dy$ – separable equation

$\int x^2 \ln x \, dx \quad \left| \begin{array}{ll} u = \ln x & du = \frac{dx}{x} \\ dv = x^2 dx & v = \frac{x^3}{3} \end{array} \right|$

$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

$\int \frac{(y+1)^2}{y} dy =$

$= \int \frac{y^2+2y+1}{y} dy$

$= \int \left(y + 2 + \frac{1}{y}\right) dy$

$= \frac{y^2}{2} + 2y + \ln|y|$

$\Rightarrow \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \frac{y^2}{2} + 2y + \ln|y|$

To find equilibrium solution:

$$\frac{dx}{dy} = \frac{1}{y \ln x}\frac{(y+1)^2}{x^2} = \frac{(y+1)^2}{y}\underbrace{\frac{1}{x^2 \ln x}}_{\neq 0}$$

– no equilibrium solution $x(y)$