First order linear ODE:

$$a_1(x)y' + a_0(x)y = f(x)$$

If $a_1(x) \neq 0 \Rightarrow$ divide by $a_1(x) \Rightarrow$

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{f(x)}{a_0(x)}$$; set: $p(x) = \frac{a_0(x)}{a_1(x)}$

and $q(x) = \frac{f(x)}{a_0(x)} \Rightarrow \quad y' + p(x)y = q(x)$

– linear equation in the standard form.

Ex 1: $y' + x^2 y = 0$ – linear equation, $p(x) = x^2$
$q(x) = 0$

Note: $y' = -x^2 y$ – separable equation as well.

Can solve: $\frac{dy}{dx} = -x^2 y \Rightarrow \frac{dy}{y} = -x^2 dx$

$$\Rightarrow \int \frac{dy}{y} = -\int x^2 dx \Rightarrow \ln|y| = -\frac{x^3}{3} + C$$

Ex 2: $xy' + 2(\sin x)y = e^x$ – linear equation

To write in standard form, divide by x:

$$y' + \underbrace{\frac{2\sin x}{x}}_{p(x)} y = \underbrace{\frac{e^x}{x}}_{q(x)}$$ – not separable.

Ex3: $y' + y = e^x$ – linear in standard form, not separable.

Actually $y' + y = 0$ – linear equation, also separable

$$\Rightarrow y' = -y \Rightarrow \frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -dx$$

$$\int \frac{dy}{y} = -\int dx \Rightarrow \ln|y| = -x + C \Rightarrow |y| = e^{-x+C}$$

$$\Rightarrow |y| = e^C e^{-x} = \underbrace{D}_{D>0} e^{-x} \Rightarrow y = \begin{cases} De^{-x} \\ -De^{-x} \end{cases} = \underbrace{E}_{\text{any real }\#} e^{-x}$$

Assume that we can write a solution of

$$y' + y = e^x$$

as $y(x) = v(x)e^{-x} \Rightarrow y'(x) = v'(x)e^{-x} - v(x)e^{-x}$

$$\Rightarrow v'e^{-x} \cancel{- ve^{-x}} \cancel{+ ve^{-x}} = e^x \Rightarrow v'e^{-x} = e^x$$

$$\Rightarrow v' = e^x e^x = e^{2x}$$ – separable equation. Solve:

$$\Rightarrow \frac{dv}{dx} = e^{2x} \Rightarrow dv = e^{2x}dx \Rightarrow \int dv = \int e^{2x}dx$$

$$\Rightarrow v = \frac{1}{2}e^{2x} + C \Rightarrow y(x) = \left(\frac{1}{2}e^{2x} + C\right)e^{-x}$$

$$\Rightarrow y = \frac{1}{2}\underbrace{e^{2x}e^{-x}}_{e^{2x-x}} + Ce^{-x} = \frac{1}{2}e^x + Ce^{-x}$$

– one parameter family of solution

Note: Ce^{-x} - one-parameter family of solutions of $y'+y=0$

$y_p = \frac{1}{2}e^x$: $y_p' = \frac{1}{2}e^x \Rightarrow y_p' + y_p = \frac{1}{2}e^x + \frac{1}{2}e^x = e^x$

$\downarrow$

solves the original ODE.

Given a linear ODE $y' + p(x)y = f(x)$, we say that this ODE is homogeneous if $f(x) \equiv 0$, otherwise the ODE is nonhomogeneous. $\Rightarrow$

In the example: $y'+y=e^x$ - nonhomogeneous equation

$y'+y=0$ - homogeneous equation

We found that the general solution of $y'+y=e^x$ is the sum of a particular solution of $y'+y=e^x$ <u>and</u> the general solution of the corresponding homogeneous equation.

How to solve $y'+p(x)y=q(x)$?

Step I: Compute the integrating factor

$$\mu(x) = e^{\int p(x)\,dx}$$

Compute $\mu' = \left(e^{\int p(x)dx}\right)' = e^{\int p(x)dx}\left(\int p(x)dx\right)' = \mu p$

Step II: Multiply the equation by $\mu(x)$

$\mu y' + \mu p y = \mu q \Rightarrow \mu y' + \mu' y = \mu q \Rightarrow (\mu y)' = \mu q$

$\Rightarrow \mu y = \int \mu q \, dx \Rightarrow y = \frac{1}{\mu} \int \mu q \, dx$

Ex: Solve $y' + y = e^x$

(1) $\mu(x) = e^{\int 1 dx} = e^x$

$(\mu y)' = \mu q \Leftrightarrow (e^x y)' = e^x \cdot e^x = e^{2x}$

$\Rightarrow e^x y = \int e^{2x} dx = \frac{1}{2} e^{2x} + c$

$y = \frac{1}{2} e^x + c e^{-x}$ ✓

Ex: $y' + xy = x$

(1) Find the integrating factor:

$$\mu(x) = e^{\int x dx} = e^{x^2/2}$$

(2) $\left(e^{x^2/2} y\right)' = x e^{x^2/2} \Rightarrow e^{x^2/2} y = \int x e^{x^2/2} dx$

$\underset{du = x dx}{\overset{u = x^2/2}{=}} \int e^u du = e^u + c \Rightarrow e^{x^2/2} + c$

$$e^{x^2/2}y = e^{x^2/2} + C \Rightarrow y = 1 + Ce^{-x^2/2}$$

Ex. $3y' + 15y = 6x$ – linear, first order ODE

Use the integrating factor: $\mu(x) = e^{\int p(x)\,dx}$ for the equation $y' + p(x)y = q(x)$; in this case the ODE is not in the normal form.

$\Rightarrow$ Divide by 3: $y' + 5y = 2x \Rightarrow$

$\mu(x) = e^{\int 5\,dx} = e^{5x} \Rightarrow$ multiply the ODE by $\mu(x)$:

$$(\mu(x)y)' = \mu(x)q(x)$$

$$\Rightarrow (e^{5x}y)' = 2xe^{x}$$

$$\Rightarrow e^{5x}y = \int 2xe^{x}\,dx \quad \left| \begin{array}{ll} u = 2x & du = 2dx \\ dv = e^{5x}dx & v = \frac{1}{5}e^{5x} \end{array} \right|$$

$$= \frac{2}{5}xe^{5x} - \frac{2}{5}\int e^{5x}\,dx = \frac{2}{5}xe^{5x} - \frac{2}{25}e^{5x} + C$$

$$\Rightarrow y = \frac{2}{5}x - \frac{2}{25} + Ce^{-5x}$$ – general solution of the ODE

Consider $y_p(x) = \frac{2}{5}x - \frac{2}{25}$ and $y_h(x) = Ce^{-5x}$

$\Rightarrow$ $y_p'(x) = \frac{2}{5}$ and $y_h'(x) = -5Ce^{-5x}$

$\Rightarrow$ $3y_p' + 15y_p = 3\cdot\frac{2}{5} + 15\left(\frac{2}{5}x - \frac{2}{25}\right) = \cancel{\frac{6}{5}} + 6x - \cancel{\frac{6}{5}} = 6x$

hence y_p is a particular solution to the nonhomogeneous ODE $3y' + 15y = 6x$

$\Rightarrow$ $3y_h' + 15y_h = 3\left(-5Ce^{-5x}\right) + 15Ce^{-5x} = 0$

hence y_h solves a homogeneous equation with the same left hand side as the original ODE:

$3y' + 15y = 0$ – in fact, y_h is a general solution of this ODE.

$\therefore$ General solution of a nonhomogeneous linear ODE is a sum of a particular solution of this ODE and a general solution of the corresponding homogeneous equation. (verify this for examples solved already)

Ex. Solve the initial value problem:

$$\begin{cases} xy' + (1+x)y = e^{-x}\sin 2x \\ y(\pi) = 1 \end{cases}$$ – linear, nonhomogeneous eq. of the 1st order

Rewrite the ODE in the normal form:

$$y' + \frac{1+x}{x}y = \frac{1}{x}e^{-x}\sin 2x$$

$$\Rightarrow \quad \mu(x) = e^{\int \frac{1+x}{x}dx} = e^{\int(\frac{1}{x}+1)dx} = e^{\ln x + x}$$

$$= e^{\ln x}e^{x} = xe^{x}$$

$$\Rightarrow \quad (xe^{x}y)' = (\cancel{\tfrac{1}{x}}\cancel{e^{-x}}\sin 2x)\cancel{x}\cancel{e^{x}} = \sin 2x$$

$$\Rightarrow \quad xe^{x}y = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$$

$$\Rightarrow \quad y = \underbrace{-\frac{\cos 2x}{xe^{x}}}_{\text{particular solution of the original ODE}} + \underbrace{\frac{C}{xe^{x}}}_{\text{general solution of the corresponding homogeneous ODE.}}$$

$$1 = y(\pi) = -\frac{\overset{1}{\cancel{\cos 2\pi}}}{\pi e^{\pi}} + \frac{C}{\pi e^{\pi}}$$

$$\pi e^{\pi} = -1 + C \Rightarrow C = \pi e^{\pi} + 1$$

$$\Rightarrow \quad y = -\frac{\cos 2x}{x e^{x}} + \frac{\pi e^{\pi} + 1}{x e^{x}}$$ – solution of the IVP.