Suppose that the function $y^2 + x^3 - x = 1$ is specified implicitly. What kind of ODE will a function like this satisfy?

Take an implicit derivative:

$$\frac{d}{dx}\left(y^2 + x^3 - x\right) = \frac{d}{dx} 1 = 0$$

$$(*) \qquad 2y\frac{dy}{dx} + 3x^2 - 1 = 0$$ – this gives us an equation for the implicit derivative, but (*) can also be thought of as an ODE

$$2yy' + 3x^2 - 1 = 0$$

that our implicit function is a solution of. This ODE is also a separable equation:

$$2y\,dy = -(3x^2-1)\,dx$$

– can solve this by integrating both sides.

Also note that, if we replace 1 by c in the definition of the implicit function, we still get (*) after taking an implicit derivative $\Rightarrow$ $y^2 + x^3 - x = c$ is a one-parameter family of solutions (or the general solution) of (*).

Let's now see what ODE the general implicit expression

$$F(x,y) = C \quad \text{would satisfy.}$$

Since y is a function of x, we can write

$$F(x, y(x)) = C \quad \text{for all } x.$$

Now take the derivative of both sides w.r.t. x (using the multidimensional chain rule):

$$\frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial x} = 0$$

– this is called the firs order <u>exact</u> ODE

Can also write it in this form:

$$\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x} dx = 0$$

To summarize: $F(x,y) = C$ is the general solution of the exact equation $\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x} dx = 0$

Now, suppose we have an equation:

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

It will be exact if there exists a function $F(x,y)$ s.t. $P(x,y) = \partial F / \partial x$ and $Q(x,y) = \partial F / \partial y$

How would we know that such function exists?
If it does $\Rightarrow$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial x \partial y}$$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, because $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$.

It turns out that the reverse is also true:

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and these partials are continuous,

then $P = \frac{\partial F}{\partial x}$ and $Q = \frac{\partial F}{\partial y}$

$\Rightarrow$ The ODE $P dx + Q dy = 0$ is exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Ex. Is the equation $x^2 dx + \cos y \, dy = 0$ exact?
If yes, find the general solution.

We have that $P(x,y) = x^2$ and $Q(x,y) = \cos y$ $\Rightarrow$

$\frac{\partial P}{\partial y} = 0$, $\frac{\partial Q}{\partial x} = 0$ so that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ – the ODE is exact.

Because the ODE is exact, there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = P(x,y) = x^2, \quad \frac{\partial F}{\partial y} = Q(x,y) = \cos y$$

To find F, integrate the first equation in x while holding y fixed:

$$F(x,y) = \int \frac{\partial F}{\partial x} dx = \int P dx = \int x^2 dx = \frac{x^3}{3} + C(y)$$

- note that C may depend on y because y is being held constant. To find $C(y)$ we take the derivative of F w.r.t. y:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3}{3} + C(y)\right) = C'(y) = Q = \cos y$$

$$\Rightarrow C'(y) = \cos y \Rightarrow C(y) = \int \cos y \, dy = \sin y$$

$$\Rightarrow F(x,y) = \frac{x^3}{3} + \sin y$$

Recall that $F(x,y) = C$ is the general solution of the exact equation $\frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial x} dx = 0 \Rightarrow$ our general solution is

$$\boxed{\frac{x^3}{3} + \sin y = C}$$

Suppose now that we have a separable equation

$$y' = f(x) g(y) \Rightarrow$$

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{dy}{g(y)} = f(x)\,dx \Rightarrow \underbrace{f(x)}_{P(x,y)}dx - \underbrace{\frac{1}{g(y)}}_{Q(x,y)}dy = 0$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(f(x)\right) = 0$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{g(y)}\right) = 0$$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$ any separable equation is also exact!

– the opposite is not true as we show in the next example.

Ex. Is the equation $(x-3y)dx - (e^y+3x)dy = 0$ exact? If yes, find the general solution.

First, note that this equation is not separable.

Here $P(x,y) = x - 3y$ and $Q(x,y) = -(e^y+3x) \Rightarrow$

$$\frac{\partial P}{\partial y} = -3$$

$$\frac{\partial Q}{\partial x} = -3$$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ – equation is exact.

$$\Rightarrow \frac{\partial F}{\partial x} = P = x - 3y,\ \frac{\partial F}{\partial y} = Q = -(e^y + 3x) \Rightarrow$$

$$F(x,y) = \int \frac{\partial F}{\partial y}dy = \int Q\,dy = -\int (e^y + 3x)\,dy$$

$$= -(e^y + 3xy) + c(x)$$

$\Rightarrow$

$$x - 3y = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left[-(e^y + 3xy) + c(x)\right] = -3y + c'(x)$$

$\Rightarrow c'(x) = x \Rightarrow c(x) = \frac{x^2}{2} \Rightarrow F(x,y) = -(e^y + 3xy) + \frac{x^2}{2}$

Thus the general solution of the ODE

$$-(e^y + 3xy) + \frac{x^2}{2} = C$$

- the solution is in the implicit form.

Ex. Solve the initial value problem

$$\begin{cases} (x-3y)\,dx - (e^y + 3x)\,dy = 0 \\ y(0) = 1 \end{cases}$$

From the previous example, the general solution is

$-(e^y + 3xy) + \frac{x^2}{2} = C \Rightarrow -(e^1 + \cancel{3 \cdot 0 \cdot 1}^{\,0}) + \cancel{\frac{0^2}{2}}^{\,0} = C \Rightarrow C = -e$

$\Rightarrow$ solution of the IVP is $-(e^y + 3xy) + \frac{x^2}{2} = -e$

Ex. Is the equation $(y\ln y - e^{-xy})\,dx + \left(\frac{1}{y} + x\ln y\right)dy = 0$ exact? If yes, find the general solution.

$$\underbrace{(y\ln y - e^{-xy})}_{P}\,dx + \underbrace{\left(\frac{1}{y} + x\ln y\right)}_{Q}\,dy = 0$$

Check: $\frac{\partial P}{\partial y} \overset{?}{=} \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(y\ln y - e^{-xy}\right) = \ln y + y\cdot\frac{1}{y} - e^{-xy}(-x)$$

$$= \ln y + 1 + xe^{-xy}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y} + x\ln y\right) = \ln y \quad \Rightarrow$$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ – the equation is not exact.

Ex. Is the equation $3xy^2dy + (x^3+y^3)dx = 0$ exact? If yes, find the general solution.

$\underbrace{3xy^2}_{Q}dy + \underbrace{(x^3+y^3)}_{P}dx = 0 \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^3+y^3) = 3y^2$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftarrow \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3xy^2) = 3y^2$

– equation is exact $\Rightarrow$ $P = \partial F/\partial x$, $Q = \partial F/\partial y$

$$\frac{\partial F}{\partial x} = x^3 + y^3, \quad \frac{\partial F}{\partial y} = 3xy^2$$

$$\Rightarrow F(x,y) = \int (x^3+y^3)dx = \frac{x^4}{4} + y^3x + C(y)$$

$$\cancel{3xy^2} = \frac{\partial F}{\partial y} = \cancel{3y^2x} + C'(y) \Rightarrow C'(y) = 0 \Rightarrow C = 0$$

$$F(x,y) = \boxed{\frac{x^4}{4} + y^3x = C}$$

Ex. Is the equation $(1+\ln x+\frac{y}{x})dx = (1-\ln x)dy$ exact? If yes, find the general solution.

$(1+\ln x+\frac{y}{x})dx = (1-\ln x)dy$

$$\underbrace{(1+\ln x+\frac{y}{x})}_{P}dx - \underbrace{(1-\ln x)}_{Q}dy = 0 \Rightarrow \frac{\partial P}{\partial y} = \frac{1}{x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftarrow \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(\ln x - 1) = \frac{1}{x}$$

– equation is exact

$$\Rightarrow \frac{\partial F}{\partial x} = 1+\ln x+\frac{y}{x}$$

$$\frac{\partial F}{\partial y} = \ln x - 1 \Rightarrow F(x,y) = \int(\ln x - 1)dy = (\ln x-1)y + C(x)$$

$$1+\ln x+\cancel{\frac{y}{x}} = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left((\ln x-1)y + C(x)\right) = \cancel{\frac{y}{x}} + C'(x)$$

$$\Rightarrow C'(x) = \ln x + 1 \Rightarrow C(x) = \int(\ln x+1)dx$$

$$= \int \ln x\,dx + \int 1\,dx = \int \ln x\,dx + x \quad \left(\begin{matrix} u=\ln x & du=\frac{dx}{x} \\ dv=dx & v=x \end{matrix}\right)$$

$$= x\ln x + x - \int \cancel{x}\frac{dx}{\cancel{x}} = x\ln x + \cancel{x} - \cancel{x} = x\ln x$$

$$\Rightarrow F(x,y) = \boxed{(\ln x-1)y + x\ln x = C}$$

Ex. Solve the initial value problem

$$\begin{cases} x\dfrac{dy}{dx} = 2xe^x - y + 6x^2 \\ y(1) = 0 \end{cases}$$

$$x\frac{dy}{dx} = 2xe^x - y + 6x^2 \Rightarrow x\,dy = (2xe^x - y + 6x^2)\,dx$$

$$\Rightarrow \underbrace{(2xe^x - y + 6x^2)}_{P}\,dx \underbrace{- x}_{Q}\,dy = 0$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial P}{\partial y} &= -1 \\ \frac{\partial Q}{\partial x} &= -1 \end{aligned}\right\} \text{are equal - exact}$$

$$\frac{\partial F}{\partial x} = 2xe^x - y + 6x^2, \quad \frac{\partial F}{\partial y} = -x$$

$$F(x,y) = \int (2xe^x - y + 6x^2)\,dx = 2\int xe^x\,dx - \int y\,dx$$

$$+ 6\int x^2\,dx = 2(xe^x - e^x) - yx + 2x^3 + C(y), \text{ since}$$

$$\int xe^x\,dx = \left| \begin{matrix} u = x & du = dx \\ dv = e^x dx & v = e^x \end{matrix} \right| = xe^x - \int e^x\,dx = xe^x - e^x$$

$$\cancel{-x} = \frac{\partial F}{\partial y} = \cancel{-x} + C'(y) \Rightarrow C'(y) = 0 \Rightarrow C(y) = 0.$$

$$\Rightarrow F(x,y) = \boxed{2(xe^x - e^x) - yx + 2x^3 = C}$$

$$\Rightarrow 2(\underbrace{1\cdot e^1 - e^1}_{0}) - \underbrace{0\cdot 1}_{0} + 2\cdot 1^3 = C \Rightarrow C = 2$$

$2(xe^x - e^x) - yx + 2x^3 = 2$ //

Note: $x\frac{dy}{dx} = 2xe^x - y + 6x^2 \Rightarrow$

$xy' + y = 2xe^x + 6x^2$ – linear equation

– can solve by using integrating factor.