$y' = \dfrac{xy+y^2}{x^2}$ – not separable, not linear. Maybe exact? Check:

$$\frac{dy}{dx} = \frac{xy+y^2}{x^2} \Rightarrow x^2 dy = (xy+y^2)dx$$

$$\Rightarrow \underbrace{(xy+y^2)}_{P}dx \underbrace{-x^2}_{Q}dy = 0$$

$$\frac{\partial P}{\partial y} = x+2y \; ; \; \frac{\partial Q}{\partial x} = -2x$$

$$\Rightarrow \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \text{ – not exact.}$$

Still want to solve:

$y' = \dfrac{\cancel{x}y}{x\cancel{{}^2}} + \dfrac{y^2}{x^2} = \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$. Try substitution:

let $v = y/x \Rightarrow y = vx \overset{\text{Product}}{\underset{\text{rule}}{\Rightarrow}} y' = v'x + v \Rightarrow$ Substitute:

$$v'x + \cancel{v} = \cancel{v} + v^2 \Rightarrow x\frac{dv}{dx} = v^2 \Rightarrow \frac{dv}{v^2} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v^2} = \int \frac{dx}{x} \Rightarrow -\frac{1}{v} = \ln|x| + C \Rightarrow -\frac{x}{y} = \ln|x| + C$$

Consider all equations of the type $y' = f\left(\frac{y}{x}\right)$ – call these homogeneous equations. Solve by using the substitution $v = \dfrac{y}{x} \Rightarrow y = vx \Rightarrow y' = v'x + vx'$

$\Rightarrow v'x + v = f(v) \Rightarrow x\dfrac{dv}{dx} = f(v) - v$ – separable ODE

$$\Rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x} \Rightarrow \int \frac{dv}{f(v)-v} = \int \frac{dx}{x} = \ln|x| + C$$

Ex: $xy' = \sqrt{x^2+y^2} + y$, $x>0$ — determine whether this equation is homogeneous; if yes, solve.

suppose that $y = vx \Rightarrow xy' = \sqrt{x^2+(vx)^2} + vx$

$$\Rightarrow xy' = \sqrt{x^2(1+v^2)} + vx = \sqrt{x^2}\sqrt{1+v^2} + xv$$

$$= |x|\sqrt{1+v^2} + xv = x\sqrt{1+v^2} + xv \Rightarrow$$

$$\cancel{x}y' = \cancel{x}\left(\sqrt{1+v^2} + v\right) \Rightarrow y' = \sqrt{1+v^2} + v$$

— no x left $\Rightarrow$ the equation is homogeneous $\Rightarrow$

$$v = \frac{y}{x},\quad y = vx \Rightarrow y' = v'x + v \xRightarrow[\text{substitution}]{\text{finish}}$$

$$v'x + \cancel{v} = \sqrt{1+v^2} + \cancel{v}$$ — separable equation;

$$x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} = \ln|x| + C \overset{x>0}{=} \ln x + C$$

Next $\displaystyle\int \frac{dv}{\sqrt{1+v^2}} \overset{\text{Trig}}{\underset{\text{subst:}}{=}}$ $\displaystyle\int \frac{\frac{1}{\cos^2 t}\,dt}{\sqrt{\underbrace{\tan^2 t + 1}_{\frac{1}{\cos^2 t}}}} = \int \frac{1/\cos^2 t}{1/\cos t}\,dt$

$v = \tan t$

$dv = \frac{1}{\cos^2 t}\,dt$

$$= \int \frac{1}{\cos t}\,dt = \int \frac{\cos t}{\cos^2 t}\,dt = \int \frac{\cos t\,dt}{1-\sin^2 t} \underset{du=\cos t\,dt}{\overset{u=\sin t}{=}}$$

$$= \int \frac{du}{1-u^2} \underset{\text{fractions}}{\overset{\text{partial}}{=}} \int \left(\frac{A}{1-u} + \frac{B}{1+u}\right) du =$$

$$= \frac{1}{2}\int \frac{du}{1-u} + \frac{1}{2}\int \frac{du}{1+u}$$

$$= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u|$$

$$= -\frac{1}{2}\ln|1-\sin t| + \frac{1}{2}\ln|1+\sin t|$$

$$= -\frac{1}{2}\ln\left|1-\frac{v}{\sqrt{1+v^2}}\right| + \frac{1}{2}\ln\left|1+\frac{v}{\sqrt{1+v^2}}\right|$$

$$-\frac{1}{2}\ln\left|1-\frac{v}{\sqrt{1+v^2}}\right| + \frac{1}{2}\ln\left|1+\frac{v}{\sqrt{1+v^2}}\right| = \ln x + C$$

$$-\frac{1}{2}\ln\left|1-\frac{y/x}{\sqrt{1+(y/x)^2}}\right| + \frac{1}{2}\ln\left|1+\frac{y/x}{\sqrt{1+(y/x)^2}}\right| = \ln x + C$$

$$\frac{A}{1-u} + \frac{B}{1+u} = \frac{1}{1-u^2}$$

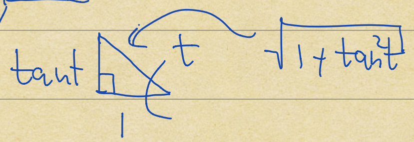
$$\frac{A(1+u)+B(1-u)}{1-u^2} = \frac{1}{1-u^2}$$

$$\iff A(1+u)+B(1-u)=1$$

$u=-1$: $2B=1$

$u=1$: $2A=1$

$\Rightarrow A=B=\frac{1}{2}$



$$\Rightarrow \sin t = \frac{\tan t}{\sqrt{1+\tan^2 t}}$$

$$\Rightarrow \sin t = \frac{v}{\sqrt{1+v^2}}$$

Bernoulli's equation:

$$y' + p(x)y = q(x)y^n$$

- this equation is not linear; can be solved using substitution: $v = y^{1-n}$

$\Rightarrow y = v^{\frac{1}{1-n}} \Rightarrow y' = \frac{1}{1-n} v^{\frac{1}{1-n}-1} v' = \frac{1}{1-n} v^{\frac{n}{1-n}} v'$

$\Rightarrow \frac{1}{1-n} v^{\frac{n}{1-n}} v' + p(x) v^{\frac{1}{1-n}} = q(x) \left(v^{\frac{1}{1-n}}\right)^n$

$\Rightarrow \frac{1}{1-n} v^{\frac{n}{1-n}} v' + p(x) v^{\frac{1}{1-n}} = q(x) v^{\frac{n}{1-n}}$

$\Rightarrow$ Multiply $(1-n) v^{-\frac{n}{1-n}}$:

$$v' + p(x)(1-n) v^{\frac{1}{1-n}} v^{-\frac{n}{1-n}} = (1-n) q(x)$$

$$v' + p(x)(1-n) v^{\underbrace{\frac{1}{1-n} - \frac{n}{1-n}}_{1}} = (1-n) q(x)$$

$\Rightarrow v' + p(x)(1-n) v = (1-n) q(x)$ – linear equation for v.

Ex. $y' = \frac{xy + y^2}{x^2} = \frac{1}{x} y + \frac{1}{x^2} y^2, \; x > 0 \quad \Rightarrow$

$$y' - \frac{1}{x} y = \frac{1}{x^2} y^2 \text{ – Bernolli's equation}$$

Use substitution: $v = y^{1-2} = y^{-1} \Rightarrow y = v^{-1}$

$\Rightarrow y' = -v^{-2} v' \Rightarrow$ substitute:

$$-v^{-2} v' - \frac{1}{x} v^{-1} = \frac{1}{x^2} (v^{-1})^2$$

$$-v^{-2}v' - \frac{1}{x}v^{-1} = \frac{1}{x^2}v^{-2} \Rightarrow$$ multiply by $-v^2$

$$v' + \frac{1}{x}v^2v^{-1} = -\frac{1}{x^2} \Rightarrow v' + \frac{1}{x}v = -\frac{1}{x^2}$$

– linear 1st order ODE.

$\Rightarrow$ Integrating factor $\mu(x) = e^{\int \frac{1}{x}dx}$

$$= e^{\ln x} = x$$

$$\Rightarrow (xv)' = -\frac{1}{x^2}\cdot x = -\frac{1}{x} \Rightarrow xv = -\int \frac{1}{x}dx$$

$$\Rightarrow xv = -\ln x + C \Rightarrow v = -\frac{1}{x}\ln x + \frac{C}{x}$$

$$\Rightarrow y = v^{-1} = \left(-\frac{1}{x}\ln x + \frac{C}{x}\right)^{-1} = \frac{x}{C - \ln x}$$

Ex. Solve the homogeneous equation

$$(x^2+y^2)dx + (x^2-xy)dy = 0$$

To check if homogeneous substitute $y = vx$:

$$(x^2 + (vx)^2)dx + (x^2 - x\cdot vx)dy = 0$$

$$(x^2 + v^2x^2)dx + (x^2 - vx^2)dy = 0 \Rightarrow$$

$$x^2[(1+v^2)dx + (1-v)dy] = 0$$

$\Rightarrow (1+v^2)dx + (1-v)dy = 0$ - no x left

$\Rightarrow$ equation is homogeneous

$\Rightarrow$ Finish substitution:

$y = vx \Rightarrow y' = v'x + v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v$

$\Rightarrow dy = x\,dv + v\,dx \Rightarrow$

$(1+v^2)dx + (1-v)(x\,dv + v\,dx) = 0$

$(1+v^2)dx + x(1-v)dv + v(1-v)dx = 0$

$(1+\cancel{v^2}+v-\cancel{v^2})dx + x(1-v)dv = 0$

$(1+v)dx + x(1-v)dv = 0$

$(1+v)dx = -x(1-v)dv = x(v-1)dv$

$\frac{dx}{x} = \frac{v-1}{v+1}dv \Rightarrow \int \frac{dx}{x} = \int \frac{v-1}{v+1}dv$

$\Rightarrow \ln|x| + c \underset{du=dv}{\overset{u=v+1}{=}} \int \frac{u-2}{u}du = \int\left(1-\frac{2}{u}\right)du$

$= \int du - 2\int \frac{1}{u}du = u - 2\ln|u|$

$\Rightarrow u - 2\ln|u| = \ln|x| + c \Rightarrow v+1-2\ln|v+1| = \ln|x| + c$

$\Rightarrow \frac{y}{x} + 1 - 2\ln\left|\frac{y}{x}+1\right| = \ln|x| + c$

Ex. Solve $x\frac{dy}{dx}+y=x^2y^2,\ x>0$ - Bernoulli's equation

Use substitution $v=y^{1-2}=\frac{1}{y} \Leftrightarrow y=\frac{1}{v}=v^{-1}$

$\Rightarrow y'=-v^{-2}v'$:

$$x(-v^{-2}v')+v^{-1}=x^2(v^{-1})^2$$

$$-xv^{-2}v'+v^{-1}=x^2v^{-2}$$ Multiply by v^2:

$-xv'+v=x^2$ - linear equation:

$\Rightarrow v'-\frac{1}{x}v=-x \Rightarrow$ integrating factor:

$$\mu(x)=e^{\int(-\frac{1}{x})dx}=e^{-\ln|x|}=e^{-\ln x}$$

$$=e^{\ln x^{-1}}=x^{-1} \Rightarrow$$

$$\left(\frac{1}{x}v\right)'=-\cancel{x}\cdot\frac{1}{\cancel{x}}=-1 \Rightarrow$$

$$\frac{1}{x}v=\int(-1)dx=-x+C \Rightarrow$$

$$v=-x^2+Cx \Rightarrow y=\frac{1}{v}=\frac{1}{Cx-x^2}$$

Ex. Solve $\begin{cases} \dfrac{dy}{dx} = (y+x)^2+3 \\ y(0)=1 \end{cases}$

use substitution:

$v = y+x \Rightarrow y = v-x \Rightarrow y' = v'-1 \Rightarrow$

$v'-1 = v^2+3 \Rightarrow v' = v^2+4$ - separable equation

$\dfrac{dv}{dx} = v^2+4 \Rightarrow \dfrac{dv}{v^2+4} = dx \Rightarrow \int \dfrac{dv}{v^2+4} = \int dx$

Then $\int \dfrac{dv}{v^2+4} = \int \dfrac{dv}{v^2+2^2} \overset{\text{table}}{=} \dfrac{1}{2}\tan^{-1}\dfrac{v}{2} \Rightarrow$

$\dfrac{1}{2}\tan^{-1}\dfrac{v}{2} = x + C \Rightarrow \tan^{-1}\dfrac{v}{2} = 2x + C$

$\Rightarrow \dfrac{v}{2} = \tan(2x+C) \Rightarrow v = 2\tan(2x+C)$

$\Rightarrow y - x = 2\tan(2x+C) \Rightarrow y = x + 2\tan(2x+C)$

Solve IVP: $1 = \cancel{0} + 2\tan(\cancel{2 \cdot 0} + C)$

$2\tan C = 1 \Rightarrow \tan C = \dfrac{1}{2} \Rightarrow C = \tan^{-1}\dfrac{1}{2}$

$\Rightarrow y = x + 2\tan\left(2x + \tan^{-1}\dfrac{1}{2}\right)$

Ex. Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$

Ex. Solve $\begin{cases} x\dfrac{dy}{dx} - (1+x)y = xy^2 \\ y(1) = 0 \end{cases}$