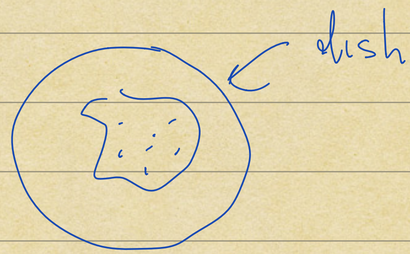


(*) Assume that there are N_0 bacteria in the dish at time 0.

(*) Find the number of bacteria in the dish at time t.

Denote the # of bacteria in the dish at time t by $N(t)$ - an unknown function, however, $N(0)=N_0$. In principle, $N(t)$ should be integer-valued, however, we can assume that $N(t)$ is real-valued if $N(t)$ is large. Know that evolution of the bacterial population will satisfy the following:

(1) $N(t)$ is $\uparrow$

(2) Each bacteria is multiplying $\Rightarrow$ expect that the rate of growth of $N(t)$ should be proportional to $N(t)$. If we suppose that the coefficient of proportionality is $k \geq 0$

$$\Rightarrow \quad N'(t) = k N(t)$$

$\Rightarrow$ Need to solve: $\begin{cases} N' = kN \leftarrow \text{linear, } 1^{st} \text{ order ODE} \\ N(0) = N_0 \end{cases}$

$$\Rightarrow N' - kN = 0 \Rightarrow \mu(t) = e^{\int(-k)dt} = e^{-kt}$$

$$\Rightarrow (e^{-kt}N)' = 0 \Rightarrow e^{-kt}N = C \Rightarrow N(t) = Ce^{kt}$$

$$\Rightarrow N_0 = N(0) = Ce^{0} = C \Rightarrow N(t) = N_0 e^{kt}$$

- population of bacteria evolves exponentially in time.

Ex. Suppose that there are initially 1000 bacteria in the dish. How many bacteria are there at $t = 10$ days?

$$\Rightarrow N(t) = 1000\, e^{kt} \quad \text{- how to find } k?$$

Suppose that we also know that there are 2000 bacteria in the dish after 1 hr. Can we use this info to find k? $\Rightarrow$ $1\text{hr} = \frac{1}{24}\text{day} \Rightarrow$

$$2000 = N\left(\tfrac{1}{24}\right) = 1000\, e^{k\cdot\frac{1}{24}}$$

$$\Rightarrow e^{k/24} = 2 \Rightarrow (e^{k})^{1/24} = 2 \Rightarrow$$

$e^{k} = 2^{24}$. We have $N(t) = 1000e^{kt}$

$$= 1000(e^{k})^{t} = 1000\,(2^{24})^{t} = 1000\cdot 2^{24t}$$

$\Rightarrow \quad N(t) = 1000 \cdot 2^{24t} \Rightarrow N(10) = 1000 \cdot 2^{240}$

– very large # !

Ex: Suppose that T_2 is the time it takes the # of bacteria to double. Want to find connection between T_2 and k. Since

$N(t) = N_0 e^{kt} \quad \Rightarrow \quad 2N_0 = N(T_2) = N_0 e^{kT_2}$

$\Rightarrow e^{kT_2} = 2 \Rightarrow (e^k)^{T_2} = 2 \Rightarrow e^k = 2^{1/T_2} \Rightarrow$

$\Rightarrow N(t) = N_0 (e^k)^t = N_0 (2^{1/T_2})^t = N_0 2^{t/T_2}$

Clearly, the linear model for bacterial growth can only be accurate when $N(t)$ is relatively small. Suppose that the dish cannot support more than N_m – maximum # of bacteria $\Rightarrow$ one way to account for this is to assume that k depends on N:

Properties of k: (1) k is positive when $N < N_m$ – the bacteria still has room to multiply

(2) When $N = N_m$ the multiplication stops $\Rightarrow$

$k(N_m) = 0$

(3) If $N > N_m \Rightarrow k(N) < 0$ – bacteria primarily dies.

What is the easiest expression for $k(N)$?

$$k = k_0 (N_m - N), \text{ where } k_0 > 0$$

– satisfies (1) – (3)

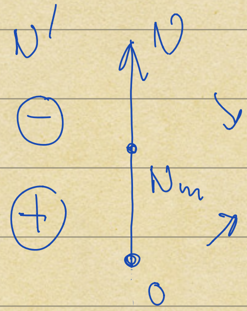
$\Rightarrow$ Modified model for bacterial growth:

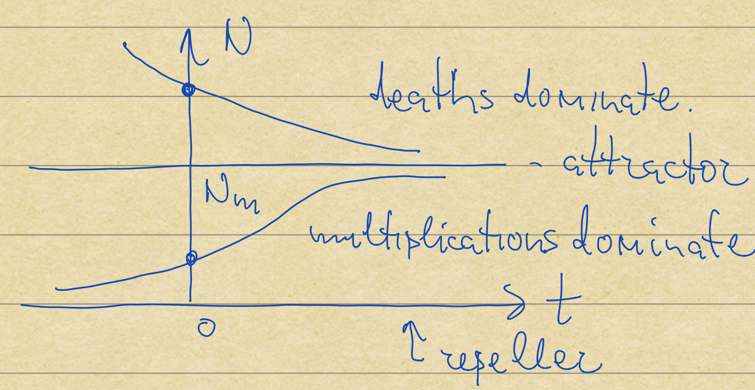
$$\begin{cases} N' = \underbrace{k_0 (N_m - N)}_{k(N)} N \\ N(0) = N_0 \end{cases}$$

– nonlinear, separable, autonomous 1st order ODE

Consider $N' = k_0 (N_m - N) N$; setting

$k_0 (N_m - N) N = 0 \Rightarrow$ equilibrium solutions are $N = 0$ and $N = N_m$





Radioactive decay:

Suppose that we have the mass m_0 of radioactive material with half-life $T_{1/2}$. What is the remaining mass after time t?

Let $m(t)$ denote the mass remaining at time t

$\Rightarrow m(0) = m_0$. The rate of decay $= m'(t)$ and

(1) $m'(t) < 0$

(2) $m'(t) = -km(t)$, $k > 0$

$\Rightarrow m' = -km$ is the linear ODE governing radioactive decay

To determine $m(t)$, solve the IVP:

$$\begin{cases} m' = -km \Rightarrow m' + km = 0 \\ m(0) = m_0 \end{cases}$$

$$\Rightarrow \mu(t) = e^{\int k\,dt} = e^{kt} \Rightarrow (e^{kt} m)' = 0 \Rightarrow$$

$$\Rightarrow e^{kt} m = C \Rightarrow m(t) = Ce^{-kt}$$

$$\Rightarrow m_0 = m(0) = Ce^{0} \Rightarrow C = m_0$$

$$\Rightarrow m(t) = m_0 e^{-kt}$$

To relate k to $T_{1/2}$: $\frac{1}{2} m_0 = m(T_{1/2}) = m_0 e^{-kT_{1/2}}$

$$\Rightarrow \frac{1}{2}\cancel{m_0} = \cancel{m_0} e^{-kT_{1/2}} \Rightarrow e^{-kT_{1/2}} = \frac{1}{2}$$

$$\Rightarrow \left(e^{-k}\right)^{T_{1/2}} = \frac{1}{2} \Rightarrow e^{-k} = \left(\frac{1}{2}\right)^{1/T_{1/2}}$$

$$\Rightarrow m(t) = m_0 e^{-kt} = m_0 \left(e^{-k}\right)^t = m_0 \left[\left(\frac{1}{2}\right)^{1/T_{1/2}}\right]^t$$

$$= m_0 \left(\frac{1}{2}\right)^{t/T_{1/2}}$$

$$\Rightarrow m(t) = m_0 \frac{1}{2^{t/T_{1/2}}}$$

Ex: Suppose we initially have 1kg of radioactive material with half-life 10 days. How much of the material is left after 12 hours?

$$\Rightarrow 12\,hrs = \frac{1}{2}\,day \Rightarrow m\left(\frac{1}{2}\right) = ?$$

$$m\left(\frac{1}{2}\right) = \frac{1}{2^{1/2/10}} = \frac{1}{2^{1/20}}.$$

Ex: Suppose we initially have 1kg of radioactive material and 300 g are left after 1 hr. How much is left after 12 hrs?

$$m(t) = m_0 e^{-kt} = e^{-kt}; \quad \text{also},$$

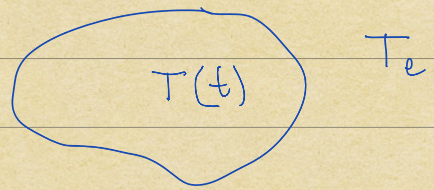
$$0.3 = m(1) = e^{-k\cdot 1} = e^{-k} \Rightarrow m(t) = e^{-kt} = \left(e^{-k}\right)^t$$

$= (0.3)^t \Rightarrow m(12) = (0.3)^{12}$ kg.

Newton's Law of Cooling:

Initial temperature:

$T(0) = T_0$



The rate of change of temperature is $T'(t)$;

Facts:

(1) If $T(t) < T_e$, then $T'(t) > 0$

(2) If $T(t) > T_e$, then $T'(t) < 0$

(3) If $T(t) = T_e$, then $T'(t) = 0$

Note: We will make an assumption that the temperature is the same throughout the body.

The simplest expression for T':

$$T' = -k(T - T_e), \quad k > 0$$

Need to solve the IVP:

$$\begin{cases} T' = -k(T - T_e) \\ T(0) = T_0 \end{cases}$$ – linear ODE of the 1st order

Solve the ODE: use the substitution

$u = T - T_e \Rightarrow u' = T' \Rightarrow u' = -ku$ - same equation as the one for radioactive decay $\Rightarrow$

$$u = u_0 e^{-kt} \underset{u = T - T_e}{\Rightarrow} T - T_e = (T_0 - T_e) e^{-kt}$$

$$\Rightarrow T = T_e + (T_0 - T_e) e^{-kt} //$$

Ex: Suppose that the temperature of the cup of tea when freshly poured is 90°C. Suppose that the cup is taken outside where the temperature is 0°C. If after 10 min. the temperature of the cup is 80°C, what is the temperature after 20 minutes?

$$T_0 = 90°C; \quad T_e = 0°C; \quad T(10) = 80°$$

$$\Rightarrow T = 0 + (90 - 0) e^{-kt} = 90 e^{-kt}$$

$$\Rightarrow 80 = T(10) = 90 e^{-k \cdot 10} = 90 (e^{-k})^{10}$$

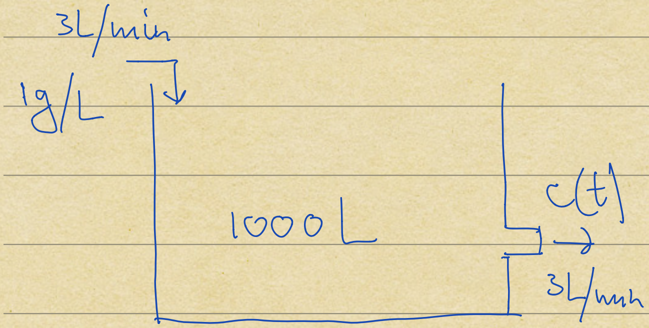
$$\Rightarrow (e^{-k})^{10} = \frac{8}{9} \Rightarrow e^{-k} = \left(\frac{8}{9}\right)^{1/10}$$

$$\Rightarrow T(t) = 90 e^{-kt} = 90 (e^{-k})^t = 90 \left(\left(\frac{8}{9}\right)^{1/10}\right)^t$$

$$= 90 \left(\frac{8}{9}\right)^{t/10}$$

$$\Rightarrow T(20) = 90\left(\frac{8}{9}\right)^{20/10} = 90\left(\frac{8}{9}\right)^2$$

$$= 10 \cdot \frac{8^2}{9} = \frac{640}{9}.$$



Suppose that a pool initially contains 1000 L of pure water. Brine solution with salt concentration of 1 g/L enters the pool at the rate of 3L/min and the well-mixed solution leaves the pool at the same rate. How much salt is in the tank at time t?

Suppose that the concentration of salt in the pool at time t is $c(t) \Rightarrow m(t) = 1000\,c(t)$ – total mass of salt inside the tank.

$$m'(t) = \text{inflow} - \text{outflow} = 3\,L/\text{min} \cdot 1\,g/L$$
$$- 3\,L/\text{min}\; c(t)\, g/L$$

$$\Rightarrow m' = 3 - 3c(t) = 3 - 3\cdot\frac{m(t)}{1000}$$

IVP for salt:

$$\begin{cases} m' = 3 - \frac{3}{1000} m \\ m(0) = 0 \end{cases}$$

$$m' + \frac{3}{1000} m = 3 \Rightarrow \mu(t) = e^{\int \frac{3}{1000} dt} = e^{\frac{3t}{1000}}$$

$$\left(e^{3t/1000} m\right)' = 3e^{\frac{3t}{1000}} \Rightarrow e^{3t/1000} m = 3 \int e^{\frac{3t}{1000}} dt$$

$$= \cancel{3} \cdot \frac{1000}{\cancel{3}} e^{3t/1000} + C \Rightarrow$$

$$m(t) = 1000 + Ce^{-3t/1000}$$

$$0 = m(0) = 1000 + Ce^{0} \Rightarrow C = -1000$$

$$\Rightarrow m(t) = 1000 - 1000e^{-3t/1000}$$