$$a_2 y'' + a_1 y' + a_0 y = f(x) \qquad (P)$$

Corresponding homogeneous eq:

$$a_2 y'' + a_1 y' + a_0 y = 0 \qquad (H)$$

Suppose that $y_h(x)$ solves (H) and $y_p(x)$ solves (P) $\Rightarrow$ then $y_h(x) + y_p(x)$ solves (P).

Ex. 1: $\quad y'' + 2y' + y = 1$

$\Rightarrow$ homogeneous equation: $y'' + 2y' + y = 0$

$\Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$

$$y_h(x) = C_1 e^{-x} + C_2 x e^{-x}$$

Need to find y_p: $\quad y_p = 1 \Rightarrow y_p' = y_p'' = 0$:

substitute $\quad 0 + 0 + 1 = 1 \Rightarrow y_p$ is a solution

$\Rightarrow$ Solution of the nonhomogenous

ODE:

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{x} + 1$$

Ex. 2: $\quad y'' + 2y' + y = x$

General solution: $y = y_h + y_p$

Corresponding homogeneous equation is the same as in the previous example

$$\Rightarrow y = c_1 e^{-x} + c_2 x e^{-x} + y_p$$

Try to guess y_p again: $y_p = x \Rightarrow$

$y_p' = 1$, $y_p'' = 0 \Rightarrow y_p'' + 2y_p' + y_p = 0 + 2\cdot 1 + x = 2 + x \neq x$

$\Rightarrow y_p = x$ is not a solution.

Try instead: $y_p = Ax + B$ where A and B are unknown constansts $\Rightarrow y_p' = A$

$\Rightarrow y_p'' = 0$. Then $y_p'' + 2y_p' + y_p = 0 + 2A + Ax + B = x$

$\Rightarrow Ax + B + 2A = x$ for all $x \Rightarrow$

$$\begin{cases} A = 1 \\ B + 2A = 0 \end{cases} \Rightarrow A = 1 \,;\; B = -2A = -2$$

$\therefore$ $y_p = x - 2$ is a particular solution

$$\Rightarrow y(x) = c_1 e^{-x} + c_2 x e^{-x} + x - 2$$

In general:

$$a_2 y'' + a_1 y' + a_0 y = b_2 x^2 + b_1 x + b_0$$

1. Find $y_h(x)$
2. Look for a particular solution of nonhomogeneous equation in the form
$$y_p(x) = Ax^2 + Bx + C$$
where A, B, C are to be determined.

– method of undetermined coefficients.

Ex 3: $\quad y'' + 2y' + y = x^3$

$$\Rightarrow y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$y_p(x) = Ax^3 + Bx^2 + Cx + D$$

$$\Rightarrow \quad y(x) = C_1 e^{-x} + C_2 x e^{-x} + Ax^3 + Bx^2 + Cx + D$$

To find A, B, C, D: $\quad y_p' = 3Ax^2 + 2Bx + C$

$$y_p'' = 6Ax + 2B$$

substitute back into the equation:

$$y_p'' + 2y_p' + y_p = \underline{6Ax} + 2B + 2\left(\underline{3Ax^2} + \underline{2Bx} + C\right)$$

$$+ \underline{Ax^3} + \underline{Bx^2} + \underline{Cx} + D = x^3$$

$$\Rightarrow Ax^3 + (6A+B)x^2 + (6A+4B+C)x + 2B + 2C + D = x^3$$

for all x $\Rightarrow \begin{cases} A = 1 \\ 6A + B = 0 \Rightarrow B = -6A \Rightarrow B = -6 \\ 6A + 4B + C = 0 \Rightarrow C = -6A - 4B = 18 \\ 2B + 2C + D = 0 \Rightarrow D = -2B - 2C = -24 \end{cases}$

$$\Rightarrow y(x) = C_1 e^{-x} + C_2 x e^{-x} + x^3 - 6x^2 + 18x - 24$$

Ex. 4: $\quad y'' + 2y' + y = e^x$

$\Rightarrow \quad y_h(x) = C_1 e^{-x} + C_2 x e^{-x}$; $\quad y_p = Ae^x$; $\quad y_p' = Ae^x$;

$y_p'' = Ae^x$;

substitute: $Ae^x + 2Ae^x + Ae^x = e^x$

$$\Rightarrow 4A\cancel{e^x} = \cancel{e^x} \Rightarrow A = \frac{1}{4} \Rightarrow y_p = \frac{1}{4}e^x$$

$$\Rightarrow \quad y(x) = C_1 e^{-x} + C_2 x e^{-x} + \frac{1}{4}e^x$$

Ex. 5 $\quad y'' + 2y' + y = xe^x$

$y_p = xe^x$? Does not work (check)

Try instead $y_p = (Ax+B)e^x \Rightarrow y_p' = Ae^x$
$+ (Ax+B)e^x = (Ax+A+B)e^x$; $y_p'' = Ae^x$
$+ (Ax+A+B)e^x = (Ax+2A+B)e^x$

$$\Rightarrow y_p'' + 2y_p' + y_p = (Ax+2A+B)e^x + 2(Ax+A+B)e^x$$
$$+ (Ax+B)e^x = e^x\left(4Ax + 4A + 4B\right) = xe^x$$

$\Rightarrow \quad 4Ax + 4A + 4B = x$ for all x

$$\begin{cases} 4A = 1 \Rightarrow A = \frac{1}{4} \\ 4A + 4B = 0 \Rightarrow B = -A = -\frac{1}{4} \end{cases} \Rightarrow y_p(x) = \frac{1}{4}(x-1)e^x$$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + \frac{1}{4}(x-1)e^x$$

Ex. 6 $\quad y'' + 3y' + 2y = e^{-x}$

$y_h(x)$: $\quad y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0$

$\Rightarrow r = -1, -2 \Rightarrow y_h(x) = C_1 e^{-x} + C_2 e^{-2x}$

$y_p(x)$: $\quad y_p(x) = Ae^{-x}$ - solves the homogeneous ODE

Should try instead:

$y_p(x) = Axe^{-x}$; $y_p'(x) = Ae^{-x} - Axe^{-x}$;

$y_p''(x) = -Ae^{-x} - Ae^{-x} + Axe^{-x} = (-2A + Ax)e^{-x}$

$\Rightarrow y_p'' + 3y_p' + 2y_p = (-2A + \cancel{Ax})e^{-x} + 3(A - \cancel{Ax})e^{-x}$

$+ 2\cancel{Ax}e^{-x} = e^{-x}A = e^{-x}$

$\Rightarrow A = 1 \Rightarrow y_p(x) = xe^{-x}$

$y(x) = C_1 e^{-x} + C_2 e^{-2x} + xe^{-x}$

Ex. 7 $\quad y'' + 2y' + y = e^{-x}$

Recall: $y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$

Need $y_p(x)$: Try $y_p = A e^{-x}$ –

this solves the homogeneous equation;

try instead $y_p = A x e^{-x}$ – again

solves the homogeneous equation;

Try now $y_p = A x^2 e^{-x}$

$\Rightarrow y_p' = 2Axe^{-x} - Ax^2 e^{-x}$

$y_p'' = 2Ae^{-x} - \underline{2Axe^{-x}} - \underline{2Axe^{-x}} + Ax^2 e^{-x}$

$\Rightarrow 2A\cancel{e^{-x}} - 4Ax\cancel{e^{-x}} + Ax^2\cancel{e^{-x}}$

$+ 2(2Ax\cancel{e^{-x}} - Ax^2\cancel{e^{-x}}) + Ax^2\cancel{e^{-x}} = \cancel{e^{-x}}$

$\Rightarrow 2A \cancel{- 4Ax} + \underline{\cancel{Ax^2}} + \cancel{4Ax} - \underline{\cancel{2Ax^2}} + \underline{\cancel{Ax^2}} = 1$

$\Rightarrow 2A = 1 \Rightarrow A = 1/2 \Rightarrow y_p = \frac{1}{2} x^2 e^{-x}$

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2} x^2 e^{-x}$$