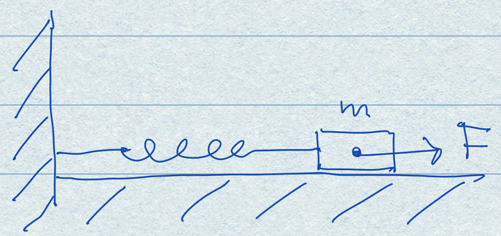


Second Newton's Law:

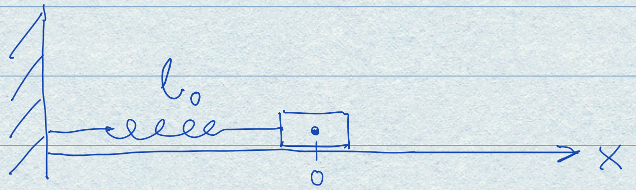
$$ma = F$$

The system is in equilibrium when the length of the spring is l_0; How would the block move after it is shifted from the equilibrium position?

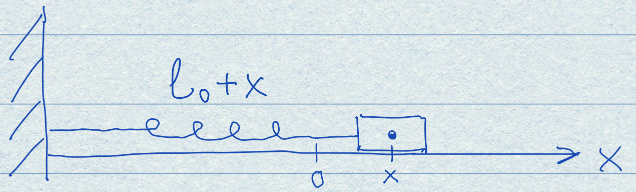
$$F = F_s + F_D$$

F_s: force due to stretching/compression of the spring

F_D: resistance/damping force



Elongation of the spring $= 0 \Rightarrow$ no force on the block



Elongation of the spring $= x \Rightarrow$ if $x > 0$, there is a force in the negative direction of x-axis and vice versa.

$\Leftarrow$ From experimental evidence, the magnitude of the force is proportional to relative elongation:

$$F_s = -\frac{\alpha x}{l_0} = -kx,$$ where $k > 0$ is a spring constant

Now, the resistance force between the block and the ground is proportional to the velocity of the block and it acts in the direction opposite to the block motion since it attempts to slow it down:

$$F_d = -\beta v, \quad \text{where } \beta \geq 0$$

$\Rightarrow$ The 2^{nd} Newton's Law is:

$$ma = F_s + F_d = -kx - \beta v$$

Now, the position of the block relative to the equilibrium length of the spring is x and x can vary with time t $\Rightarrow$

$$\text{position} = x(t)$$

$\Rightarrow$ velocity $v = \dot{x}(t)$ (Notation: $\dot{x}(t) = \frac{dx}{dt}$)

$\Rightarrow$ acceleration $a = \dot{v}(t) = \ddot{x}(t)$

and the 2^{nd} Newton's Law becomes

$$m\ddot{x} = -kx - \beta\dot{x} \quad \text{or}$$

$$m\ddot{x} + \beta\dot{x} + kx = 0$$ – 2nd order linear homogeneous ODE w/constant coefficients

To fully determine the position of the block at any time, we also need to know its initial position and velocity

$\Rightarrow$ position of the block at any time can be determined by solving the IVP:

$$\begin{cases} m\ddot{x} + \beta\dot{x} + kx = 0 \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

I. Assume that there is no friction, i.e.,

$\beta = 0 \Rightarrow$ IVP becomes:

$$\begin{cases} m\ddot{x} + kx = 0 \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

To solve: characteristic equation

$$mr^2 + k = 0 \Rightarrow r^2 = -\frac{k}{m} < 0$$

Notation:
Set $\omega^2 = \frac{k}{m}$

$$\Rightarrow r^2 = -\omega^2 \Rightarrow r = \pm i\omega$$

$$\Rightarrow x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

Therefore, as expected, without friction the position of the block will oscillate relative to the equilibrium, never stopping.

We can use the initial data to determine c_1 and c_2:

$$\dot{x}(t) = -\omega c_1 \sin\omega t + \omega c_2 \cos\omega t$$

$$\Rightarrow \quad x_0 = x(0) = c_1 \; ; \quad v_0 = \dot{x}(0) = \omega c_2 \Rightarrow c_2 = \frac{v_0}{\omega}$$

$$\Rightarrow \quad x(t) = x_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t$$

Notice that we can also rewrite $x(t)$ as follows:

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t$$

$$= \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \left[\frac{x_0}{\sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}} \cos\omega t + \frac{v_0/\omega}{\sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}} \sin\omega t \right]$$

Now, because

$$\left(\frac{x_0}{\sqrt{x_0^2+\frac{v_0^2}{\omega^2}}}\right)^2 + \left(\frac{v_0/\omega}{\sqrt{x_0^2+\frac{v_0^2}{\omega^2}}}\right)^2 = 1 \qquad \text{(check)}$$

we can find φ such that

$$\cos\varphi = \frac{v_0/\omega}{\sqrt{x_0^2+\frac{v_0^2}{\omega^2}}}, \quad \sin\varphi = \frac{x_0}{\sqrt{x_0^2+\frac{v_0^2}{\omega^2}}} \Rightarrow \tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{x_0\omega}{v_0}$$

$$\Downarrow$$

$$\varphi = \tan^{-1}\frac{x_0\omega}{v_0}$$

Also, denote

$$A = \sqrt{x_0^2+\frac{v_0^2}{\omega^2}}, \quad \text{then}$$

$$x(t) = A\left(\sin\varphi\cos\omega t + \cos\varphi\sin\omega t\right)$$

$\Rightarrow$ $x(t) = A\sin(\omega t+\varphi)$, because

$$\sin a\cos b + \cos a\sin b = \sin(a+b)$$

– trig. identity

Therefore, $x(t)$ oscillates between $\pm A$ – we call A the amplitude of oscillations; φ is called the phase;

Because the sine function has a period 2π, the motion of the block is periodic w/period $T = \frac{2\pi}{\omega}$: indeed $\omega(t+T)+\varphi = \omega(t+\frac{2\pi}{\omega})+\varphi$

$$= (\omega t + \varphi) + 2\pi \implies$$

$$\sin(\omega(t+T)+\varphi) = \sin(\omega t + \varphi + 2\pi) = \sin(\omega t + \varphi)$$

$\implies$ the block repeats its motion after time T.

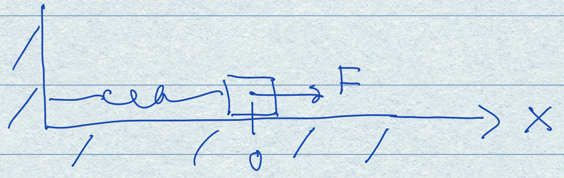
The number of times the block will repeat its motion in one unit of time is then given by $f = \frac{1}{T}$ – this is called the frequency of oscillations.

Ex. 1: Suppose that a block of mass 2kg is attached to a wall with a spring and that the block can be moved on the floor in the direction perpendicular to the wall. A test of the spring reveals that stretching the spring by 10 cm requires the force of 0.8N. If the block is moved away from the wall by 20 cm and then released from rest, determine the position of the block at any time. What is the amplitude, the phase, the period and the frequency of motion of the block if there is no resistance between the block and the floor?

(1) The force required to stretch the spring by $10cm = 0.1\,m$ is $0.8N \Rightarrow$ the spring force compensating the applied force is $k \cdot 0.1 \Rightarrow$

$$k \cdot 0.1m = 0.8N \Rightarrow k = 8\frac{N}{m}$$

(2)



$$ma = F_s + F_d$$
$$\Downarrow$$
$$ma = -kx - \beta v$$
$$\Downarrow$$
$$\underset{2}{m}\ddot{x} = -\underset{8}{k}x - \underset{0\text{ – no resistance}}{\beta}\dot{x}$$

$$\begin{cases} \ddot{x} + 4x = 0 \Leftarrow 2\ddot{x} + 8x = 0 \Leftarrow \\ x(0) = 0.2 \leftarrow 20cm = 0.2m \\ \dot{x}(0) = 0 \leftarrow \text{block is released from rest.} \end{cases}$$

Solve the ODE: $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$\Rightarrow x(t) = c_1 \cos 2t + c_2 \sin 2t$$
$$\Rightarrow \dot{x}(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$$

$$\left. \begin{aligned} 0.2 &= x(0) = c_1 \cdot 1 + c_2 \cdot 0 \\ 0 &= \dot{x}(0) = -2c_1 \cdot 0 + 2c_2 \cdot 1 \end{aligned} \right\} \Rightarrow c_1 = 0.2,\ c_2 = 0$$

Therefore: $x(t) = 0.2\cos 2t = 0.2\sin\left(2t + \frac{\pi}{2}\right)$

because of the trig identity: $\sin\left(a + \frac{\pi}{2}\right) = \cos a$

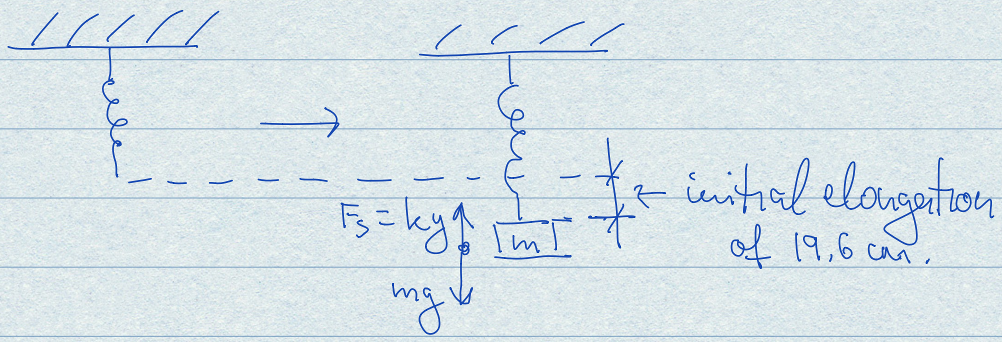
Then $x(t) = \underbrace{0.2}_{\text{amplitude } A} \sin\left(\underbrace{2}_{\omega} t + \underbrace{\tfrac{\pi}{2}}_{\text{phase } \varphi}\right)$

$\Rightarrow \quad T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ - period

$f = \dfrac{1}{T} = \dfrac{1}{\pi}$ - frequency

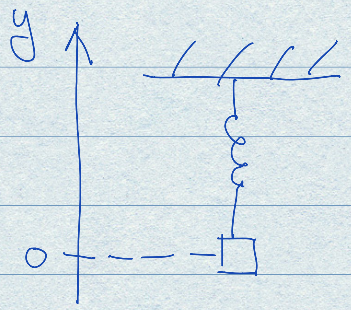
Ex. 2 Suppose that once the mass of 160g is attached to the end of the spring hanging from the ceiling, the spring is elongated by 19.6 cm. Suppose that there is no air resistace or other sources of friction. If the mass is then pulled down by 4 cm and then released with the upward velocity of 1 m/s, find the position of the block at any time t.

(1)



$\Rightarrow \quad k \cdot 0.196 = 0.16 \cdot 9.8 \quad \Rightarrow \quad k = 8\,\dfrac{N}{m}$

(2) Now that the spring force compensates gravity in equilibrium we can neglect gravity in the rest of this problem and consider the location of the block when it stops moving as the equilibrium of the spring/mass system:



No resistance $\Rightarrow \beta = 0$

$m\ddot{y} = -ky$

$\Rightarrow 0.16\ddot{y} = -8y$

$$\Rightarrow \begin{cases} \ddot{y} + 50y = 0 \\ y(0) = -0.04\,m = -4\,cm \\ \dot{y}(0) = 1\,m/s \end{cases}$$

$r^2 + 50 = 0 \Rightarrow r = \pm\sqrt{50}\,i$

$y(t) = c_1 \cos\sqrt{50}\,t + c_2 \sin\sqrt{50}\,t$

$\dot{y}(t) = -\sqrt{50}\,c_1 \sin\sqrt{50}\,t + \sqrt{50}\,c_2 \cos\sqrt{50}\,t$

$-0.04 = y(0) = c_1$

$1 = \dot{y}(0) = \sqrt{50}\,c_2 \Rightarrow c_2 = \frac{1}{\sqrt{50}}$

$\Rightarrow y(t) = -0.04\cos\sqrt{50}\,t + \frac{1}{\sqrt{50}}\sin\sqrt{50}\,t$

$$\Rightarrow A = \sqrt{0.04^2 + \left(\frac{1}{\sqrt{50}}\right)^2} = \sqrt{\left(\frac{4}{100}\right)^2 + \frac{1}{50}} = \sqrt{\frac{1}{25^2} + \frac{1}{50}}$$

$$= \sqrt{1/25}\sqrt{\frac{1}{25} + \frac{1}{2}} = \frac{1}{5}\sqrt{\frac{27}{50}} = \frac{3}{5}\sqrt{\frac{3}{50}}\ m$$

– the amplitude; period $T = \frac{2\pi}{\sqrt{50}}$ s.

Motion in presence of friction:

$$m\ddot{x} + \beta\dot{x} + kx = 0$$

$\Rightarrow$ Characteristic equation:

$$mr^2 + \beta r + k = 0$$

Divide by m:

$$r^2 + \frac{\beta}{m} r + \frac{k}{m} = 0$$

Denote: $\gamma = \frac{\beta}{2m}$ and recall that $\omega^2 = \frac{k}{m}$ $\Rightarrow$

$$r^2 + 2\gamma r + \omega^2 = 0 \quad \Rightarrow$$

$$r = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

and we have to consider three cases:

(1) $\gamma > \omega$ $\Rightarrow$ characteristic equation has two real roots

$$r_1 = -\gamma - \sqrt{\gamma^2 - \omega^2} \quad \text{and} \quad r_2 = -\gamma + \sqrt{\gamma^2 - \omega^2}$$

Clearly, $r_1 < 0$ and so is r_2 since $\sqrt{\gamma^2 - \omega^2} < \gamma$

The solution to the ODE is

$$x(t) = c_1 e^{-(\gamma+\sqrt{\gamma^2-\omega^2})t} + c_2 e^{(-\gamma+\sqrt{\gamma^2-\omega^2})t}$$

and $x(t) \to 0$ as $t \to \infty$ as expected - the motion will eventually cease due to friction.

(2) $\gamma = \omega \Rightarrow$ characteristic equation has one solution, $r = -\gamma$. Then

$$x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t}$$

Here $x(t) \to 0$ with $t \to \infty$ as well.

(3) $\gamma < \omega \Rightarrow \gamma^2 - \omega^2 < 0 \Rightarrow \sqrt{\gamma^2-\omega^2} = \sqrt{-(\omega^2-\gamma^2)}$
$= \pm i\sqrt{\omega^2-\gamma^2}$

$$\Rightarrow r = -\gamma \pm i\sqrt{\omega^2-\gamma^2}$$

$$\Rightarrow x(t) = e^{-\gamma t}\left(c_1 \sin\sqrt{\omega^2-\gamma^2}\,t + c_2 \cos\sqrt{\omega^2-\gamma^2}\,t\right)$$

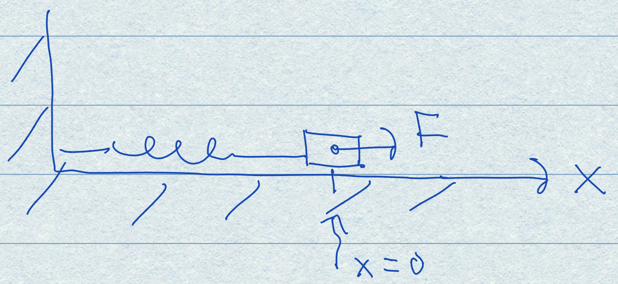
$\Rightarrow \lim_{t\to\infty} x(t) = 0$ because the exponential factor goes to 0 while sin and cos oscillate between -1 and 1.

In the three cases above, the resistance always eventually stops the motion. The strength of friction is measured by γ: It is the weakest in the case (3) and the solution in this case is still able to oscillate. This case corresponds to a spring-mass system that is underdamped.

In the cases (1) and (2) there is at most one pass through the equilibrium before the mass stops moving — these cases are known as overdamped and critically damped, respectively.

Ex. 3: Suppose that a block of mass 2kg is attached to a wall with a spring and that the block can be moved on the floor in the direction perpendicular to the floor. A test of the spring reveals that stretching the spring by 10 cm requires the force of 0.8N. Suppose, in addition, that the resistance to the motion is $F_d = -10V$.

If the block is moved away from the wall by 30 cm and then released from rest, determine the position of the block at any time. Classify the motion according to its damping strength.



The spring data is the same as in Ex. 1

$\Rightarrow k = 8\ N/m$

$\Rightarrow F = -8x - 10\dot{x}$

$\Rightarrow$ ma = F becomes $2\ddot{x} = -8x - 10\dot{x}$ or

$$\begin{cases} 2\ddot{x} + 10\dot{x} + 8x = 0 \\ x(0) = 0.3 \\ \dot{x}(0) = 0 \end{cases}$$

Divide the equation by 2: $\ddot{x} + 5\dot{x} + 4x = 0$

$\Rightarrow$ characteristic equation: $r^2 + 5r + 4 = 0$

$\Rightarrow (r+1)(r+4) = 0 \Rightarrow r = -1, -4$ $\therefore$ two real roots means that the system is overdamped

$$x(t) = c_1 e^{-t} + c_2 e^{-4t} \Rightarrow$$

$$\dot{x}(t) = -c_1 e^{-t} - 4c_2 e^{-4t}$$

$\Rightarrow$ $0.2 = x(0) = c_1 + c_2 \Rightarrow -4c_2 + c_2 = 0.3 \Rightarrow c_2 = -0.1$

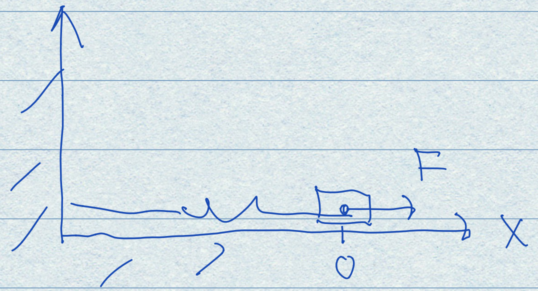
$\Uparrow$ $\Downarrow$

$0 = \dot{x}(0) = -c_1 - 4c_2 \Rightarrow c_1 = -4c_2 \Rightarrow c_1 = 0.4$

$\therefore x(t) = 0.4e^{-t} - 0.1e^{-4t}$

Driven motion: Suppose that we have the same spring-mass system as in the Ex. 1 but now assume that an external force $F(t) = 8\cos\alpha t$ is applied to the mass. If the mass is initially moved by 20 cm toward the wall and then released from rest, determine the position of the mass at any $t > 0$.

As in example 1, $k = 8\ N/m$, $\beta = 0\ \frac{N\cdot s}{m}$



$$F = -8x + 8\cos\alpha t$$
$$\Downarrow$$
$$ma = F$$
$$\Downarrow$$

$$\begin{cases} \ddot{x} + 4x = 8\cos\alpha t \\ x(0) = -0.2 \\ \dot{x}(0) = 0 \end{cases} \Leftarrow 2\ddot{x} = -8x + 8\cos\alpha t$$

Now, need to solve a nonhomogeneous ODE:

$r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow x_h(t) = c_1\cos 2t + c_2\sin 2t$

Use undetermined coefficients to find a particular solution:

$$x_p = A\cos\alpha t + B\sin\alpha t$$

$$\dot{x}_p = -\alpha A\sin\alpha t + \alpha B\cos\alpha t$$

$$\ddot{x}_p = -\alpha^2 A\cos\alpha t - \alpha^2 B\sin\alpha t$$

$$\Rightarrow \quad \ddot{x}_p + 4x_p = -\alpha^2 A\cos\alpha t - \alpha^2 B\sin\alpha t$$
$$+4(A\cos\alpha t + B\sin\alpha t) = 8\cos\alpha t$$

$$\Rightarrow \quad A(4-\alpha^2)\cos\alpha t + B(4-\alpha^2)\sin\alpha t = 8\cos\alpha t$$

$$\Rightarrow \quad A(4-\alpha^2) = 8, \quad B(4-\alpha^2) = 0$$

$$\Rightarrow \quad A = \frac{8}{4-\alpha^2}, \quad B = 0$$

$$\Rightarrow \quad x_p(t) = \frac{8}{4-\alpha^2}\cos\alpha t$$

$$\Rightarrow \quad x(t) = c_1\cos 2t + c_2\sin 2t + \frac{8}{4-\alpha^2}\cos\alpha t$$

$$\dot{x}(t) = -2c_1\sin 2t + 2c_2\cos 2t - \frac{8\alpha}{4-\alpha^2}\sin\alpha t$$

$$-0.2 = x(0) = C_1 + \frac{8}{4-\alpha^2} \Rightarrow C_1 = -\left(0.2 + \frac{8}{4-\alpha^2}\right)$$

$$0 = \dot{x}(0) = 2C_2 \Rightarrow C_2 = 0$$

$$\Rightarrow x(t) = -\left(0.2 + \frac{8}{4-\alpha^2}\right)\cos 2t + \frac{8}{4-\alpha^2}\cos \alpha t$$

Notice that for the applied force of the fixed magnitude if we change the frequency α of force oscillations, then the amplitude of $x(t)$ grows larger as $\alpha \to 2$. In fact, the solution above is not even valid when $\alpha = 2$.

To fix this problem, look for a particular solution of

$$\ddot{x} + 4x = 8\cos 2t, \quad \text{where } \alpha = 2.$$

We already know that $x_h(t) = C_1\cos 2t + C_2 \sin 2t$ then $x_p = A\cos 2t + B\sin 2t$ solves the homogeneous equation and we need to set

$$x_p = At\cos 2t + Bt\sin 2t \Rightarrow$$

$$\dot{x}_p = A\cos 2t + B\sin 2t - 2At\sin 2t + 2Bt\cos 2t$$

$$\ddot{x}_p = -4A\sin 2t + 4B\cos 2t - 4At\cos 2t - 4Bt\sin 2t$$

$$\Rightarrow \ddot{x}_p + 4x_p = -4A\sin 2t + 4B\cos 2t - 4At\cos 2t - 4Bt\sin 2t$$

$$+ 4(At\cos 2t + Bt\sin 2t) = 8\cos 2t$$

$$(-4A - \cancel{4Bt} + \cancel{4Bt})\sin 2t + (4B - \cancel{4At} + \cancel{4At})\cos 2t = 8\cos 2t$$

$$\Rightarrow 4B = 8, \quad -4A = 0 \quad \Rightarrow A = 0, B = 2$$

$$\Rightarrow \quad x_p(t) = 2t\sin 2t \quad \Rightarrow$$

$$x(t) = c_1\cos 2t + c_2\sin 2t + 2t\sin 2t$$

$$\dot{x}(t) = -2c_1\sin 2t + 2c_2\cos 2t + 2\sin 2t$$
$$+ 4t\cos 2t$$

$$-0.2 = x(0) = c_1 \quad \Rightarrow c_1 = -0.2$$

$$0 = \dot{x}(0) = 2c_2 \quad \Rightarrow c_2 = 0$$

$$x(t) = -0.2\cos 2t + 2t\sin 2t$$

- this solution still oscillates, but its amplitude $\uparrow \infty$ - resonance.