$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

Second order homogeneous C.E. equation:

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0 , \quad x > 0$$

If $y = x^k \to y' = k x^{k-1} \to y'' = k(k-1) x^{k-2}$

$$a_2 x^2 k(k-1) x^{k-2} + a_1 x k x^{k-1} + a_0 x^k = 0$$

$$\Rightarrow \quad a_2 k(k-1) x^k + a_1 k x^k + a_0 x^k = 0$$

Now divide by x^k:

$$a_2 k(k-1) + a_1 k + a_0 = 0$$

- characteristic equation for C.E.

$$\Rightarrow \quad a_2 k^2 - a_2 k + a_1 k + a_0 = 0$$

$$\Rightarrow \quad a_2 k^2 + (a_1 - a_2) k + a_0 = 0$$

Choices: (1) two real roots k_1, k_2

$\Rightarrow$ x^{k_1} and x^{k_2} are linearly independent solutions of C.E.

$\Rightarrow \quad y = c_1 x^{k_1} + c_2 x^{k_2}$

(2) one real root: $\quad k$

$\Rightarrow \quad x^k$ solves the C.E.

Use method of reduction of order to show that the second solution is:

$$x^k \ln x$$

$$\Rightarrow y(x) = c_1 x^k + c_2 x^k \ln x$$

(3) Two complex conjugate solutions:

$k = \alpha \pm i\beta \Rightarrow x^{\alpha+i\beta}$ and $x^{\alpha-i\beta}$ are complex solutions of C.E.:

$$\begin{aligned} y &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= c_1 x^{\alpha} x^{i\beta} + c_2 x^{\alpha} x^{-i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \end{aligned}$$

Consider $\quad c_1 x^{i\beta} + c_2 x^{-i\beta} = c_1 (e^{\ln x})^{i\beta} + c_2 (e^{\ln x})^{-i\beta}$

$= c_1 e^{i\beta \ln x} + c_2 e^{-i\beta \ln x} \qquad (*)$

Now, recall that $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta \Rightarrow$

add: $e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

Subtract: $e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$$\Rightarrow \boxed{\begin{aligned} &\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta} = \cos\theta \\ &\frac{1}{2i}e^{i\theta} - \frac{1}{2i}e^{-i\theta} = \sin\theta \end{aligned}}$$

Choose $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$ in $(*)$:

$$\frac{1}{2}e^{i\beta\ln x} + \frac{1}{2}e^{-i\beta\ln x} = \cos(\beta\ln x)$$

Choose $c_1 = \frac{1}{2i}$, $c_2 = -\frac{1}{2i}$ in $(*)$

$$\frac{1}{2i}e^{i\beta\ln x} - \frac{1}{2i}e^{-i\beta\ln x} = \sin(\beta\ln x)$$

$$\Rightarrow \left.\begin{aligned} &x^{\alpha}\cos(\beta\ln x) \\ &x^{\alpha}\sin(\beta\ln x) \end{aligned}\right\} - \text{are lin. ind. solutions of C.E.}$$

$$\Rightarrow y = x^{\alpha}\left(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)\right)$$

Ex 1:

$$x^2y'' + xy' + 4y = 0, \quad x > 0$$

Plug in $y = x^k \to y' = kx^{k-1} \to y'' = k(k-1)x^{k-2}$

$$\Rightarrow \cancel{x^2}k(k-1)\cancel{x^{k-2}} + \cancel{x}k\cancel{x^{k-1}} + 4\cancel{x^k} = 0$$

$$k(k-1) + k + 4 = 0 \Rightarrow$$ characteristic eq.

$$k^2 - k + k + 4 = 0 \Rightarrow k^2 + 4 = 0$$

$$\Rightarrow k = \sqrt{-4} = \pm 2i \Rightarrow \alpha = 0, \beta = 2$$

$$\Rightarrow y = C_1\cos(2\ln x) + C_2\sin(2\ln x)$$

Ex. 2:

$$\begin{cases} x^2y'' + 5xy' + 4y = 0, \quad x > 0 \\ y(1) = 1 \\ y'(1) = 0 \end{cases}$$

Characteristic equation:

$$k(k-1) + 5k + 4 = 0 \Rightarrow$$

$$k^2 - k + 5k + 4 = 0 \Rightarrow k^2 + 4k + 4 = 0$$

$$(k+2)^2 = 0 \Rightarrow k = -2$$

$$y(x) = c_1 x^{-2} + c_2 x^{-2} \ln x$$

$$y'(x) = -2c_1 x^{-3} + c_2\left(-2x^{-3}\ln x + x^{-2}x^{-1}\right)$$

$$1 = y(1) = c_1 + c_2 \ln 1 = c_1$$

$$0 = y'(1) = -2 + c_2\left(-2\cancelto{0}{\ln 1} + 1\right) = -2 + c_2$$

$$\Rightarrow c_2 = 2$$

$$y(x) = x^{-2} + 2x^{-2}\ln x$$

<u>Ex3:</u>

$$x^2 y'' - 5xy' + 8y = 0$$

Characteristic equation: $k(k-1) - 5k + 8 = 0$

$$k^2 - 6k + 8 = 0$$

$$(k-4)(k-2) = 0 \Rightarrow k = 2, 4$$

$$\Rightarrow y(x) = c_1 x^2 + c_2 x^4$$

Ex. 4

$$x^2 y'' - 5xy' + 8y = x$$

$\Rightarrow$ $y(x) = y_h(x) + y_p(x)$ where

$y_h(x) = c_1 x^2 + c_2 x^4$ (see the previous example)

To find y_p, either use undetermined coefficients or variation of parameters:

(a) U.C.: $y_p = Ax + B \Rightarrow y_p' = A,\ y_p'' = 0 \overset{\text{substitute}}{\Longrightarrow}$

$$x^2 \cdot 0 - 5x(A) + 8(Ax+B) = x$$

$$-5Ax + 8Ax + 8B = x \quad \text{or} \quad 3Ax + 8B = x$$

for all x

$$\Rightarrow \begin{cases} 3A = 1 \\ 8B = 0 \end{cases} \Rightarrow A = \frac{1}{3},\ B = 0$$

$$\Rightarrow y_p = \frac{1}{3}x$$

$\Rightarrow$ general solution: $y = c_1 x^2 + c_2 x^4 + \frac{x}{3}$

(b) Variation of parameters: First, write the ODE in the standard form to ensure that the formulas are valid:

$$y'' - \frac{5}{x}y' + \frac{8}{x^2}y = \frac{1}{x}$$

then set

$$y_p = u_1 x^2 + u_2 x^4,$$ where

$$u_1' = -\frac{f y_2}{W(y_1,y_2)} = -\frac{x^{-1}\cdot x^4}{W(x^2,x^4)} = -\frac{x^3}{2x^5} = -\frac{1}{2}x^{-2}$$

$$u_1' = \frac{f y_1}{W(y_1,y_2)} = \frac{x^{-1}\cdot x^2}{W(x^2,x^4)} = \frac{x}{2x^5} = \frac{1}{2}x^{-4}$$

$$W(x^2,x^4) = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = 4x^5 - 2x^5 = 2x^5$$

$$\Rightarrow \quad u_1 = -\int \frac{1}{2}x^{-2}\,dx = \frac{1}{2x} \; ; \; u_2 = \int \frac{1}{2}x^{-4} = -\frac{1}{6}x^{-3}$$

$$\Rightarrow \quad y_p = \frac{1}{2x}\cdot x^2 - \frac{1}{6x^3}\cdot x^4 = \frac{1}{2}x - \frac{1}{6}x = \frac{1}{3}x \; //$$

Ex. 5: Find a particular solution to

$$x^2 y'' - 5xy' + 8y = x^2 \quad , \quad x > 0$$

(a) Undetermined coefficients:

We already know that $y = x^2$ solves the homogeneous version of the ODE. Similar to the equations with constant coefficients, the approach is to multiply the function by a factor (x is the case of eq-s with constant coefficients). For C.E. the factor is $\ln x$;

Thus, instead of $y_p = Ax^2 + Bx + C$, we use

$$y_p = Ax^2\ln x + Bx + C$$

Then $y_p' = A\left(2x\ln x + x^2\cdot\frac{1}{x}\right) + B = A(2x\ln x + x) + B$

$$y_p'' = A\left(2\ln x + 2x\cdot\frac{1}{x} + 1\right) = A(2\ln x + 3)$$

Substitute into the nonhomogeneous ODE:

$$x^2 A(2\ln x + 3) - 5x\left(A(2x\ln x + x) + B\right)$$
$$+ 8\left(Ax^2\ln x + Bx + C\right) = x^2$$

$\Rightarrow$

$$\cancel{2Ax^2\ln x} + 3Ax^2 - \cancel{10Ax^2\ln x} - 5Ax^2 - 5Bx$$
$$+ \cancel{8Ax^2\ln x} + 8Bx + 8C = x^2$$

$\Rightarrow$ $(3A - 5A)x^2 + (8B - 5B)x + 8C = x^2$

$$-2Ax^2 + 3Bx + 8C = x^2 \text{ for all } x$$

$\Rightarrow$ $\begin{cases} -2A = 1 \\ 3B = 0 \\ 8C = 0 \end{cases}$ $\Rightarrow$ $A = -\frac{1}{2}$, $B = C = 0$

$$\Rightarrow y_p(x) = -\frac{1}{2}x^2\ln x$$

(b) Variation of parameters: since

$y_h = c_1 x^2 + c_2 x^4$, assume

$y_p = u_1 x^2 + u_2 x^4$.

The equation in standard form looks like:

$$y'' - \frac{5}{x} y' + \frac{8}{x^2} y = 1 \implies$$

$$u_1' = -\frac{f y_2}{W(y_1, y_2)} = -\frac{1 \cdot x^4}{2x^5} = -\frac{1}{2x}$$

same W as in the previous example

$$u_2' = \frac{f y_1}{W(y_1, y_2)} = \frac{1 \cdot x^2}{2x^5} = \frac{1}{2x^3}$$

$$\implies \quad u_1 = -\int \frac{1}{2x} dx = -\frac{1}{2} \ln|x| = -\frac{1}{2} \ln x$$

$$u_2 = \int \frac{1}{2x^3} dx = \frac{1}{2} \frac{x^{-2}}{-2} = -\frac{1}{4x^2}$$

$$\implies y_p = \left(-\frac{1}{2} \ln x\right) x^2 - \frac{1}{4x^2} x^4 = -\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$$

$$\implies y = c_1 x^2 + c_2 x^4 - \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$$

$$= \left(c_1 - \frac{1}{4}\right) x^2 + c_2 x^4 - \frac{1}{2} x^2 \ln x$$

$= c_1 x^2 + c_2 x^4 - \frac{1}{2} x^2 \ln x$ - gives the same particular solution as u.c.