$a_2 y'' + a_1 y' + a_0 y = 0$, a_0, a_1, a_2 are constant

$\Rightarrow$ characteristic equation:

$$a_2 r^2 + a_1 r + a_0 = 0$$

Three choices: (1) Two real roots: r_1, r_2

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

(2) One real root: r

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

(3) Two complex conjugate roots: $r = \alpha \pm i\beta$

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

Ex. 1:

$$y'' + 3y' + 2y = 0$$

$\Rightarrow$ $r^2 + 3r + 2 = 0$ $\Rightarrow$ $(r+1)(r+2) = 0$

$\Rightarrow$ $r_1 = -1$, $r_2 = -2$:

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}$$

Ex. 2 :

$$y'' - 10y' + 25y = 0$$

$$\Rightarrow \quad r^2 - 10r + 25 = 0 \quad \Rightarrow \quad (r-5)^2 = 0$$

$$\Rightarrow \quad r = 5 : \qquad y(x) = c_1 e^{5x} + c_2 x e^{5x}$$

Ex 3:

$$y'' - 10y' + 26y = 0$$

$$\Rightarrow \quad r^2 - 10r + 26 = 0 \quad \Rightarrow$$

$$r = \frac{10 \pm \sqrt{100 - 26 \cdot 4}}{2} = \frac{10 \pm \sqrt{-4}}{2}$$

$$= \frac{10 \pm 2i}{2} = 5 \pm i$$

$$y(x) = e^{5x}(c_1 \cos x + c_2 \sin x)$$

Ex. 4:

$$\begin{cases} y'' + 5y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$$r^2 + 5r + 4 = 0 \quad \Rightarrow \quad (r+1)(r+4) = 0$$

$$\Rightarrow \quad r = -1 \text{ and } r = -4$$

$y(x) = c_1 e^{-x} + c_2 e^{-4x}$; $y'(x) = -c_1 e^{-x} - 4c_2 e^{-4x}$

$1 = y(0) = c_1 + c_2$; $0 = y'(0) = -c_1 - 4c_2$

$$\Downarrow \qquad\qquad \Downarrow$$

$$-4c_2 + c_2 = 1 \quad \Leftarrow \quad c_1 = -4c_2$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$-3c_2 = 1$$

$$c_2 = -\frac{1}{3} \quad \Rightarrow \quad c_1 = -4 \cdot \left(-\frac{1}{3}\right) = \frac{4}{3}$$

Solution to IVP:

$$y(x) = \frac{4}{3} e^{-x} - \frac{1}{3} e^{-4x}$$

Ex.5: $y^{(IV)} - y = 0 \Rightarrow r^4 - 1 = 0$

$$\Rightarrow (r^2-1)(r^2+1) = 0$$

$$(r-1)(r+1)(r-i)(r+i) = 0$$

$$\Rightarrow \quad r = -1, \ r = 1, \ r = -i, \ r = i$$

$$y(x) = c_1 e^{-x} + c_2 e^{x} + e^{0 \cdot x}(c_3 \cos x + c_4 \sin x)$$

$$= c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x$$