In the next set of lectures we will discuss yet another way of solving linear equations with constant coefficients, but this will require additional background.

We introduce a function of functions defined via the following formula:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

for those fuctions of t for which this integral converges (i.e., it is finite).

Note that you plug in a function $f(t)$ as an input and integrate in t assuming that s is a constant. Then the value of the integral depends on s; we denote the resulting function $F(s)$.

We thus defined what is called the

Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\}$$

- input function of t and get a function of s as the output.

<u>Note:</u> Because we integrate from 0 to ∞, only the part of f defined over $t>0$ is important here.

<u>Ex 1:</u>

Suppose $f(t) = e^t \Rightarrow$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^t\} = \int_0^\infty e^{-st} e^{t}\, dt$$

$$\overset{\text{s is constant}}{=} \int_0^\infty e^{(1-s)t}\, dt = \lim_{p\to\infty} \int_0^p e^{(1-s)t}\, dt$$

$$= \lim_{p\to\infty} \left(\frac{e^{(1-s)t}}{1-s} \Big|_0^p \right) = \lim_{p\to\infty} \left(\frac{e^{(1-s)p}}{1-s} - \frac{1}{1-s} \right)$$

$$= \frac{1}{1-s} \left(\lim_{p\to\infty} e^{(1-s)p} \right) - \frac{1}{1-s}$$

$$\lim_{p\to\infty} e^{(1-s)p} = \begin{cases} 0, & s>1 \\ \infty, & s<1 \end{cases} \Rightarrow$$ the integral converges only if $s>1$

$$\Rightarrow \mathcal{L}\{e^t\} = 0 - \frac{1}{1-s} = \frac{1}{s-1} \quad \text{if } s>1$$

Ex. 2: Now, let's try the general case

$f(t) = e^{at}$, where a is a constant

Then:

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt$$

$$\overset{s \text{ is constant}}{=} \int_0^\infty e^{(a-s)t}\,dt = \lim_{p\to\infty} \int_0^p e^{(a-s)t}\,dt$$

$$= \lim_{p\to\infty} \left(\frac{e^{(a-s)t}}{a-s} \Big|_0^p \right) = \lim_{p\to\infty} \left(\frac{e^{(a-s)p}}{a-s} - \frac{1}{a-s} \right)$$

$$= \frac{1}{a-s} \left(\lim_{p\to\infty} e^{(a-s)p} \right) - \frac{1}{a-s}$$

$$\lim_{p\to\infty} e^{(a-s)p} = \begin{cases} 0, & s>a \\ \infty, & s<a \end{cases} \Rightarrow \text{the integral converges only if } s>a$$

$$\Rightarrow \mathcal{L}\{e^{at}\} = 0 - \frac{1}{a-s} = \frac{1}{s-a} \quad \text{if } s>a.$$

Ex. 3: $\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\,dt = \lim_{p\to\infty} \int_0^p e^{-st}\,dt$

$$= \lim_{p\to\infty}\left(-\frac{1}{s}e^{-st}\Big|_0^p\right) = \lim_{p\to\infty}\left(\frac{1}{s}-\frac{1}{s}e^{-sp}\right)$$

$$= \frac{1}{s} - \frac{1}{s}\lim_{p\to\infty} e^{-sp} = \frac{1}{s} - \frac{1}{s}\begin{cases} 0, & \text{if } s>0 \\ \infty, & \text{if } s<0 \end{cases}$$

$$\Rightarrow \mathcal{L}\{1\} = \frac{1}{s} \quad \text{if } s>0$$

Ex. 4: $\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$

$$= \lim_{p\to\infty}\int_0^p t^n e^{-st}\,dt$$

Need to evaluate $\int_0^p t^n e^{-st}\,dt$ ... use parts:

$$\int_0^p t^n e^{-st}\,dt = \begin{vmatrix} u=t^n & du = nt^{n-1}dt \\ dv = e^{-st}dt & v = -\frac{1}{s}e^{-st} \end{vmatrix}$$

$$= -\frac{t^n}{s}e^{-st}\Big|_0^p + \frac{n}{s}\int_0^p t^{n-1}e^{-st}\,dt$$

$$= -\frac{p^n}{s}e^{-sp} + \frac{n}{s}\int_0^p t^{n-1}e^{-st}\,dt$$

Substituting this back into the limit:

$$\mathcal{L}\{t^n\} = \lim_{p\to\infty}\left(-\frac{p^n}{s}e^{-sp} + \frac{n}{s}\int_0^p t^{n-1}e^{-st}\,dt\right)$$

$$= -\frac{1}{s}\lim_{p\to\infty} p^n e^{-sp} + \frac{n}{s}\lim_{p\to\infty}\int_0^p t^{n-1}e^{-st}\,dt$$

$$= -\frac{1}{s}\lim_{p\to\infty} p^n e^{-sp} + \frac{n}{s}\underbrace{\int_0^\infty t^{n-1}e^{-st}\,dt}_{\mathcal{L}\{t^{n-1}\}}$$

$= \infty$ if $s<0$

$\downarrow s>0$

$$\lim_{p\to\infty}\frac{p^n}{e^{sp}} \overset{L.H.}{=} \lim_{p\to\infty}\frac{np^{n-1}}{se^{sp}} \underset{n \text{ times}}{\overset{\text{repeat}}{=}} \lim_{p\to\infty}\frac{\text{const}}{e^{sp}} = 0$$

$$\Rightarrow \mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\} \quad \text{for any } n \text{ and } s>0$$

Then: $\mathcal{L}\{t\} \overset{n=1}{=} \frac{1}{s}\mathcal{L}\{t^0\} = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$

$$\mathcal{L}\{t^2\} \overset{n=2}{=} \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2\cdot 1}{s^3}$$

$$\mathcal{L}\{t^3\} \overset{n=3}{=} \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s}\cdot\frac{2\cdot 1}{s^3} = \frac{1\cdot 2\cdot 3}{s^4}$$

$$\Downarrow$$

$$\mathcal{L}\{t^n\} = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)n}{s^{n+1}} = \frac{n!}{s^{n+1}} \quad \text{if } s>0$$

Some additional properties:

(1) $\mathcal{L}\{cf(t)\} = \int_0^\infty cf(t)e^{-st}\,dt = c\int_0^\infty f(t)e^{-st}\,dt = c\mathcal{L}\{f(t)\}$

(2) $\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty (f(t)+g(t))e^{-st}\,dt$

$= \int_0^\infty (f(t)e^{-st} + g(t)e^{-st})\,dt = \int_0^\infty f(t)e^{-st}\,dt$

$+ \int_0^\infty g(t)e^{-st}\,dt = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

We conclude that Laplace Transform is linear.

Ex: $\mathcal{L}\{2e^{-t}+t^4\} = 2\mathcal{L}\{e^{-t}\} + \mathcal{L}\{t^4\}$

$$= 2\cdot\frac{1}{s-(-1)} + \frac{4!}{s^5} = \frac{2}{s+1} + \frac{4!}{s^5}$$

Additional transforms:

$$\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\frac{e^{iat}+e^{-iat}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{iat}\} + \frac{1}{2}\mathcal{L}\{e^{-iat}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{1}{2}\,\frac{s+\cancel{ia}+s-\cancel{ia}}{(s-ia)(s+ia)}$$

$$= \frac{1}{2}\,\frac{2s}{s^2-(ia)^2} = \frac{s}{s^2-i^2a^2} = \frac{s}{s^2+a^2}$$

Likewise:

$$\mathcal{L}\{\sin at\} = \mathcal{L}\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\}$$

$$= \frac{1}{2i}\mathcal{L}\{e^{iat}\} - \frac{1}{2i}\mathcal{L}\{e^{-iat}\}$$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{1}{2i}\,\frac{s+ia-s+ia}{(s-ia)(s+ia)}$$

$$= \frac{1}{2i}\,\frac{2ia}{s^2-(ia)^2} = \frac{a}{s^2-i^2a^2} = \frac{a}{s^2+a^2}$$

Next:

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{at}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{1}{2}\,\frac{s+\cancel{a}+s-\cancel{a}}{(s-a)(s+a)}$$

$$= \frac{1}{2}\,\frac{2s}{s^2-a^2} = \frac{s}{s^2-a^2}$$

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{at}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\,\frac{\cancel{s}+a-\cancel{s}+a}{(s-a)(s+a)}$$

$$= \frac{1}{2}\,\frac{2a}{s^2-a^2} = \frac{a}{s^2-a^2}.$$

Ex:

$$\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{t^2}e^{-st}\,dt = \int_0^\infty e^{t^2-st}\,dt = \infty$$

for <u>any</u> value of s: as $t \to \infty$, $t^2 - st \to \infty$

for any $s \Rightarrow e^{t^2-st} \to \infty$ as $t \to \infty$ for any

$s \Rightarrow \int_0^\infty e^{t^2-st}\,dt \to \infty$ $\therefore$ Laplace transform of e^{t^2} is undefined $\Rightarrow$ $f(t)$ should not grow "too fast" as $t \to \infty$ for the Laplace transform of $f(t)$ to make sense.

We have the following table of Laplace Transforms we have computed so far:

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$c_1 g(t) + c_2 h(t)$	$c_1 G(s) + c_2 H(s)$

Ex. $\mathcal{L}\{t^3 + 4(t+1)^2 + 2e^{1-t} + 4\cos 3t\}$

Use the table:

$= \mathcal{L}\{t^3\} + 4\mathcal{L}\{(t+1)^2\} + 2\mathcal{L}\{e^{1-t}\} + 4\mathcal{L}\{\cos 3t\}$

$= \frac{3!}{s^4} + 4\mathcal{L}\{t^2+2t+1\} + 2e\mathcal{L}\{e^{-t}\} + \frac{4s}{s^2+9}$

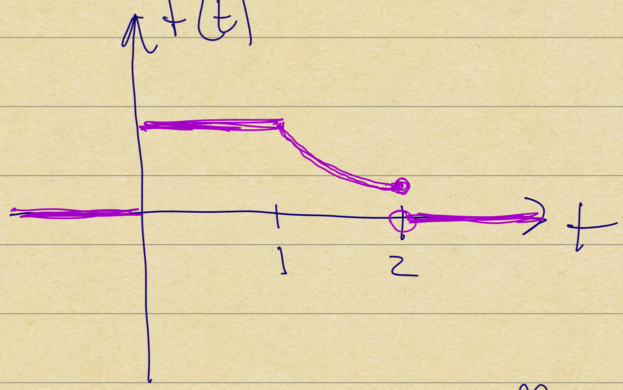
$= \frac{3!}{s^4} + \frac{2e}{s+1} + \frac{4s}{s^2+9} + 4\mathcal{L}\{t^2\} + 8\mathcal{L}\{t\} + 4\mathcal{L}\{1\}$

$= \frac{3!}{s^4} + \frac{2e}{s+1} + \frac{4s}{s^2+9} + \frac{4\cdot 2!}{s^3} + \frac{8}{s^2} + \frac{4}{s}$

Ex! Find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0, & t<0 \\ 1, & t\in[0,1] \\ e^{1-t}, & t\in[1,2] \\ 0, & t>2 \end{cases}$$

The function is piecewise-defined — cannot use the table, need to use the definition of L.T. instead.



$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^1 1 \cdot e^{-st} dt + \int_1^2 e^{1-t} e^{-st} dt$$

$$= \int_0^1 e^{-st} dt + \int_1^2 e\, e^{-(s+1)t} dt = \int_0^1 e^{-st} dt$$

$$+ e \int_1^2 e^{-(s+1)t} dt = \left(-\frac{1}{s} e^{-st}\right)\Big|_0^1 + \left(-\frac{e}{s+1} e^{-(s+1)t}\right)\Big|_1^2$$

$$= -\frac{1}{s} e^{-s} + \frac{1}{s} - \frac{e}{s+1} e^{-2(s+1)} + \frac{e}{s+1} e^{-(s+1)}$$

Note: $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \mathcal{L}\{f(t)\}$

$$= \lim_{s\to\infty} \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty f(t) \left(\lim_{s\to\infty} e^{-st}\right) dt = \int_0^\infty f(t) \cdot 0\, dt = 0$$

$\therefore$ all transformed functions $\to 0$ as $s \to \infty$

Inverse Transform: It turns out that L.T. is invertible, that is, if $F(s) = \mathcal{L}\{f(t)\}$, then there is no other function of t, say $g(t)$ such that $\mathcal{L}\{g(t)\} = F(s)$. It follows that we can define the inverse transform via

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{if} \quad \mathcal{L}\{f(t)\} = F(s)$$

Appealing to the table of Laplace transforms above, we can immideately construct the table of inverse transforms:

$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^n}$	$t^{n-1}/(n-1)!$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2+a^2}$	$\sin at$

$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{a}{s^2-a^2}$	$\sinh at$
$c_1 G(s) + c_2 H(s)$	$c_1 g(t) + c_2 h(t)$

How to obtain the second line in the table?

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = f(t), \text{ such that}$$

$\mathcal{L}\{f(t)\} = \frac{1}{s^n}$ – this looks similar to the RHS on the second line in the table of forward transforms:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow$$

$$\mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^n} \Rightarrow \frac{1}{(n-1)!}\mathcal{L}\{t^{n-1}\} = \frac{1}{s^n}$$

$$\Rightarrow \mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}\right\} = \frac{1}{s^n} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} = e^{-2t}$

Ex: $\mathcal{L}^{-1}\left\{\frac{3}{s-4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = 3e^{4t}$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^2+4}\right\}$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin 2t$$

Ex: $\mathcal{L}^{-1}\left\{\frac{6s+4}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{6s+4}{s^2+9}\right\}$

$$= 6\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{s^2+9}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = 6\cos 3t + \frac{4}{3}\sin 3t$$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{A}{s+1} + \frac{B}{s+2}\right\} = A\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= Ae^{-t} + Be^{-2t}$$

Using partial fractions to find A and B:

$$\frac{1}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)+B(s+1)}{s^2+3s+2} \Rightarrow$$

$A(s+2) + B(s+1) = 1$: $s=-2 \Rightarrow B(-1) = 1 \Rightarrow B = -1$

$s=-1 \Rightarrow A\cdot 1 = 1 = A = 1$

$$\Rightarrow \mathcal{L}^{-1}\left\{ \frac{1}{s^2+3s+2} \right\} = e^{-t} - e^{-2t}$$

Ex: $$\mathcal{L}^{-1}\left\{ \frac{3}{s^2(s-1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{A+Bs}{s^2} + \frac{C}{s-1} \right\}$$

$$= A\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} + B\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} + C\mathcal{L}^{-1}\left\{ \frac{1}{s-1} \right\}$$

$$= A\frac{t^{2-1}}{(2-1)!} + B\frac{t^{1-1}}{(1-1)!} + Ce^{t} = At + B + Ce^{-t}$$

Partial fractions: $$\frac{A+Bs}{s^2} + \frac{C}{s-1} = \frac{3}{s^2(s-1)}$$

$$\Rightarrow (A+Bs)(s-1) + Cs^2 = 3$$

$$\Rightarrow Bs^2 + As - Bs - A + Cs^2 = 3 \quad \text{for all } s$$

$$\Rightarrow (B+C)s^2 + (A-B)s - A = 3 \quad \text{for all } s$$

$$\Rightarrow \begin{cases} B+C=0 \\ A-B=0 \\ -A=3 \end{cases} \Rightarrow A=-3,\ B=-3,\ C=3$$

$$\mathcal{L}^{-1}\left\{ \frac{3}{s^2(s-1)} \right\} = -3t - 3 + 3e^{-t}$$