Recall that $\mathcal{L}\{y(t)\} = \int_0^\infty y(t)e^{-st}\,dt = Y(s)$

Want to compute $\mathcal{L}\{y'(t)\} = \int_0^\infty y'(t)e^{-st}\,dt$

$$= \left| \begin{array}{ll} u = e^{-st} & du = -se^{-st}\,dt \\ dv = y'(t)\,dt & v = y(t) \end{array} \right| = e^{-st}y(t)\Big|_0^\infty$$

$$- \int_0^\infty y(t)(-s)e^{-st}\,dt = \lim_{p\to\infty}\left(e^{-st}y(t)\Big|_0^p\right)$$

$$+ s\int_0^\infty y(t)e^{-st}\,dt = sY(s) + \lim_{p\to\infty}\left(e^{-sp}y(p)\right.$$

$$\left.- y(0)\right) = sY(s) + \cancelto{0}{\left(\lim_{p\to\infty} e^{-sp}y(p)\right)} - y(0)$$

$$\Rightarrow \mathcal{L}\{y'(t)\} = sY(s) - y(0) = s\mathcal{L}\{y(t)\} - y(0)$$

$\Rightarrow$ Want to compute now:

$$\mathcal{L}\{y''(t)\} = \mathcal{L}\{(y'(t))'\} = s\mathcal{L}\{y'(t)\} - y'(0)$$

$$= s(sY(s) - y(0)) - y'(0)$$

$$= s^2Y(s) - sy(0) - y'(0)$$

$$\Rightarrow \mathcal{L}\{y'''(t)\} = \mathcal{L}\{(y''(t))'\} = s\mathcal{L}\{y''(t)\} - y''(0)$$

$$= s\left(s^2 Y(s) - sy(0) - y'(0)\right) - y''(0)$$

$$= s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

$$\vdots$$

$$\mathcal{L}\{y^{(n)}(t)\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - s y^{(n-2)}(0) - y^{(n-1)}(0)$$

Ex: $\begin{cases} y'' - 3y' + 2y = t \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$

Take the Laplace transform of both sides of the equation:

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{t\}$$

$$\Rightarrow \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 3\left(sY(s) - y(0)\right) + 2Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - s - 2 - 3\left(sY(s) - 1\right) + 2Y(s) = \frac{1}{s^2}$$

- transformed ODE is an algebraic equation; solve this for $Y(s)$:

$$(s^2 - 3s + 2)\,Y = \frac{1}{s^2} + s + 2 - 3 = \frac{1}{s^2} + s - 1$$

$$\Rightarrow \quad Y(s) = \frac{\frac{1}{s^2} + s - 1}{s^2 - 3s + 2} = \frac{s^3 - s^2 + 1}{s^2(s^2 - 3s + 2)}$$

Now want to find $y(t)$ from $Y(s)$ $\Rightarrow$ use inverse Laplace transform:

Use partial fractions:

$$\frac{s^3 - s^2 + 1}{s^2(s^2 - 3s + 2)} = \frac{s^3 - s^2 + 1}{s^2(s-1)(s-2)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{Cs + D}{s^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s} + \frac{D}{s^2}$$

$\Rightarrow$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s} + \frac{D}{s^2}\right\}$$

$$= A\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + B\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + C\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + D\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= Ae^{t} + Be^{2t} + C + Dt$$

$$\frac{s^3-s^2+1}{s^2(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s} + \frac{D}{s^2}$$

$$= \frac{A(s-2)s^2 + B(s-1)s^2 + C(s-1)(s-2)s + D(s-1)(s-2)}{(s-1)(s-2)s^2}$$

$\Rightarrow$ $s^3-s^2+1 = A(s-2)s^2 + B(s-1)s^2 + C(s-1)(s-2)s$

$+ D(s-1)(s-2)$ for all s.

$s=1$: $1 = -A \Rightarrow A = -1$

$s=2$: $8-4+1 = 4B \Rightarrow B = \frac{5}{4}$

$s=0$: $1 = 2D \Rightarrow D = \frac{1}{2}$

$s=-1$: $-1-1+1 = -(-1-2) + \frac{5}{4}(-2) + C(-1-1)$

$* (-1-2)(-1) + \frac{1}{2}(-1-1)(-1-2)$

$\Rightarrow -1 = -6C + 3 - \frac{5}{2} + 3$

$\Rightarrow -6C = -1 - 6 + \frac{5}{2} = -\frac{9}{2}$

$\Rightarrow C = \frac{3}{4}$

$$y(t) = -e^t + \frac{5}{4}e^{2t} + \frac{3}{4} + \frac{1}{2}t$$

Check:

$$y(0) = -1 + \frac{5}{4} + \frac{3}{4} = 1 \quad \checkmark$$

Ex. Solve $\begin{cases} y'' + 5y' + 6y = 10\sin x \\ y(0) = 2 \\ y'(0) = -6 \end{cases}$

using the Laplace transform. Let $Y(s) = \mathcal{L}\{y(t)\}$ and take the Laplace transform of both sides of the ODE:

$$\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{10\sin x\} = 10\,\mathcal{L}\{\sin x\} \overset{\text{Table}}{=} \frac{10}{s^2+1}$$

$$\shortparallel$$

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = s^2Y(s) - sy(0) - y'(0)$$

$$+ 5\left(sY(s) - y(0)\right) + 6Y(s) = \frac{10}{s^2+1}$$

$$\Rightarrow s^2Y(s) - 2s + 6 + 5\left(sY(s) - 2\right) + 6Y(s) = \frac{10}{s^2+1}$$

$$\Rightarrow (s^2 + 5s + 6)Y = \frac{10}{s^2+1} + 2s - 6 + 10 = \frac{10 + (s^2+1)(2s+4)}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{10 + (s^2+1)(2s+4)}{(s^2+5s+6)(s^2+1)} = \frac{10 + (s^2+1)(2s+4)}{(s+2)(s+3)(s^2+1)}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1}$$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s+2} + \frac{B}{s+3} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1}\right\}$

$= A\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + C\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + D\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$\xRightarrow[\text{of } \mathcal{L}\{f\}]{\text{Table}} y(t) = Ae^{-2t} + Be^{-3t} + C\cos t + D\sin t.$

Now find the coefficients:

$$\frac{10 + (s^2+1)(2s+4)}{(s^2+5s+6)(s^2+1)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1} \Leftrightarrow$$

$$\Rightarrow A(s+3)(s^2+1) + B(s+2)(s^2+1) + (Cs+D)(s^2+5s+6)$$
$$= 10 + (s^2+1)(2s+4)$$

(1) $s = -3$: $(-3+2)(9+1)B = 10 + 10(2\cdot(-3)+4)$

$\Rightarrow B = 1$

(2) $s = -2$: $A \cdot 1 \cdot 5 = 10 \Rightarrow A = 2$

(3) $s = 0$: $3A + 2B + 6D = 10 + 4 = 14$

(4) $s = 1$: $8A + 6B + 12C + 12D = 22$

$A = 2,\ B = 1 \Rightarrow$

$6D = 14 - 3A - 2B = 14 - 3\cdot 2 - 2\cdot 1 = 6 \Rightarrow D = 1$

$12C = 22 - 8A - 6B - 12D = 22 - 16 - 6 - 12 = -12$

$\Rightarrow \quad C = -1$

$$y(t) = 2e^{-2t} + e^{-3t} - \cos t + \sin t.$$

Check: $y(0) = 2 + 1 - 1 + 0 = 2 \quad \checkmark$

$y'(t) = -4e^{-2t} - 3e^{-3t} + \sin t + \cos t$

$\Rightarrow y'(0) = -4 - 3 + 1 = -6 \quad \checkmark$