First Translation Theorem:

Suppose we need to find the Laplace Transform of $e^{at}f(t)$ ... is the outcome related to the transform of $f(t)$? Compute:

$$\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{at}f(t)e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a) = \mathcal{L}\{f(t)\}(s-a),$$

that is, if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

$\therefore$ Multiplication of the original function by an exponential e^{at} results in the translation of the transformed function by a along the s-axis. This is the first translation theorem.

Now, what happens if we translate $f(t)$ by a instead? What would happen to the transformed function?

Remember that the formula for the Laplace

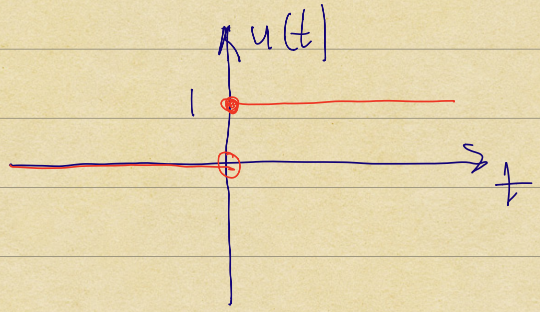
Transform only takes into account the part of $f(t)$ corresponding to $t > 0$, so we can always replace $f(t)$, $t < 0$ by 0. In other words,

$$f(t), \quad t \in (-\infty, \infty) \qquad \text{and} \qquad \tilde{f}(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

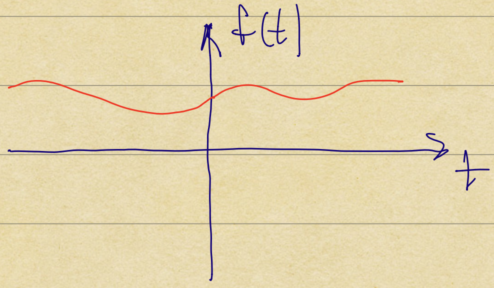
are exactly the same from the point of view of the Laplace Transform. From now on, assume that $f(t) = 0$ if $t < 0$. To accomplish this, let

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

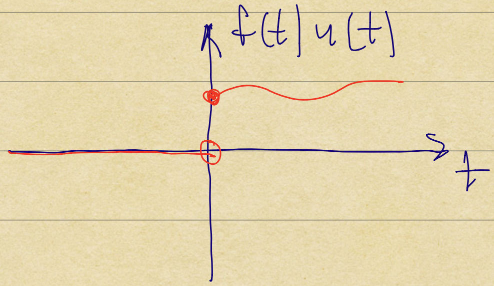
be a unit step function:



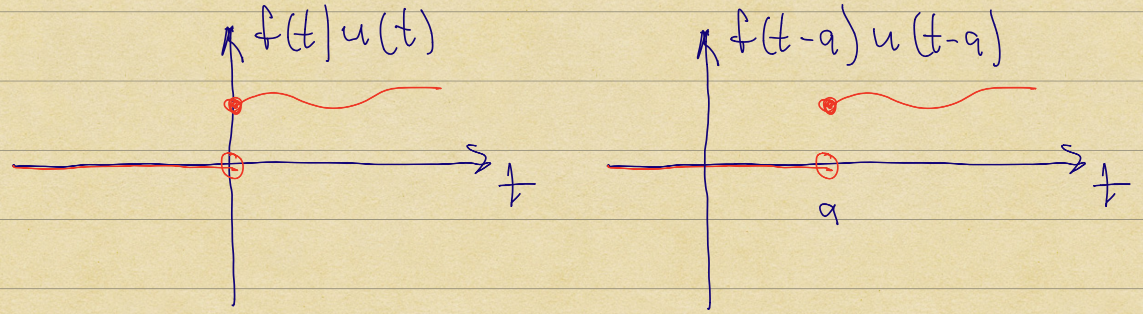
Then



$\Rightarrow$



so that translating by a along t-axis gives:



Now compute

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)e^{-st}\,dt \underset{dp=dt}{\overset{p=t-a}{=}} \int_0^\infty f(p)\,e^{-s(a+p)}\,dp$$

$$= \int_0^\infty f(p)\,e^{-as}e^{-sp}\,dp = e^{-as}\int_0^\infty f(p)e^{-sp}\,dp = e^{-as}\mathcal{L}\{f(t)\}$$

$$\Rightarrow \quad \mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

$\Rightarrow$ cutting of the part of f on the negative side of t-axis, translating by a, and taking the Laplace Transform, multiplies the transform of f by the exponential factor e^{-as}: this is the Second Translation Theorem.

Thus we have two new entries in the table of Laplace transforms:

	Original	Transform
1TT	$e^{at} f(t)$	$F(s-a)$
2TT	$f(t-a)\,u(t-a)$	$e^{-as} F(s)$

and two new entries in the table of inverse Laplace transforms:

	Transform	Original
1TT	$F(s-a)$	$e^{at} f(t)$
2TT	$e^{-as} F(s)$	$f(t-a)\,u(t-a)$

These can also be expressed in the following way:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{F(s)\}(t-a)\,u(t-a)$$

Examples:

(1) $\mathcal{L}\{e^{2t}t\} = \mathcal{L}\{t\}(s-2) = \frac{1}{(s-2)^2}$

(2) $\mathcal{L}\{e^{-t}\cos t\} = \mathcal{L}\{\cos t\}(s+1) = \dfrac{s+1}{(s+1)^2+1}$

(3) $\mathcal{L}\{\sin(t-\pi)u(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin t\}$

$$= e^{-\pi s}\frac{1}{s^2+1} = \frac{e^{-\pi s}}{s^2+1}$$

(4) $\mathcal{L}\{e^{t-1}(t+1)\} = \mathcal{L}\{e^{-1}e^{t}(t+1)\} = e^{-1}\big(\mathcal{L}\{te^{t}\}$

$$+\mathcal{L}\{e^{t}\}\big) = e^{-1}\left(\mathcal{L}\{t\}(s-1) + \frac{1}{s-1}\right)$$

$$= e^{-1}\left(\frac{1}{(s-1)^2} + \frac{1}{s-1}\right)$$

(5) $\mathcal{L}\{\cos t\, u(t-2\pi)\} = \mathcal{L}\{\cos(t-2\pi)u(t-2\pi)\}$

$$= e^{-2\pi s}\mathcal{L}\{\cos t\} = \frac{se^{-2\pi s}}{s^2+1}$$

(6) $\mathcal{L}\{\cos t\, u(t-\frac{\pi}{2})\} = \mathcal{L}\{\cos((t-\frac{\pi}{2})+\frac{\pi}{2})u(t-\frac{\pi}{2})\}$

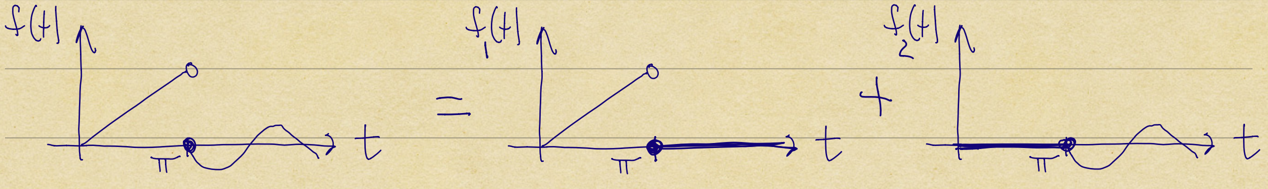
$= \mathcal{L}\{-\sin(t-\frac{\pi}{2})u(t-\frac{\pi}{2})\}$

trig identity:

$\cos(a+\frac{\pi}{2}) = -\sin a$

$= -\mathcal{L}\{\sin t\}e^{-\frac{\pi s}{2}} = -\dfrac{e^{-\frac{\pi s}{2}}}{s^2+1}$

(7) $f(t) = \begin{cases} t, & t\in[0,\pi) \\ \sin t, & t\geq\pi \end{cases}$



$$\Rightarrow \quad f(t) = f_1(t) + f_2(t) \quad , \text{ where}$$

$$f_1(t) = \begin{cases} t, & t \in [0,\pi] \\ 0, & t \geq \pi \end{cases} = t \begin{cases} 1, & t \in [0,\pi] \\ 0, & t \geq \pi \end{cases}$$

$$= t\,(1 - u(t-\pi))$$

$$f_2(t) = \begin{cases} 0, & t \in [0,\pi] \\ \sin t, & t \geq \pi \end{cases} = (\sin t) \begin{cases} 0, & t \in [0,\pi] \\ 1, & t \geq \pi \end{cases}$$

$$= (\sin t)\, u(t-\pi)$$

$$\Rightarrow \quad \mathcal{L}\{f(t)\} = \mathcal{L}\{f_1(t) + f_2(t)\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\}$$

$$= \mathcal{L}\{t\,u(t-\pi)\} + \mathcal{L}\{(\sin t)\,u(t-\pi)\} = \mathcal{L}\{((t-\pi)+\pi)\,u(t-\pi)\}$$

$$+ \mathcal{L}\{\sin((t-\pi)+\pi)\,u(t-\pi)\} \overset{2TT}{=} \mathcal{L}\{t+\pi\}\,e^{-\pi s}$$

$$+ \mathcal{L}\{\underbrace{\sin(t+\pi)}_{=-\sin t}\}\,e^{-\pi s} = e^{-\pi s}\left(\mathcal{L}\{t\} + \pi\mathcal{L}\{1\} - \mathcal{L}\{\sin t\}\right)$$

$$= e^{-\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s} - \frac{1}{s^2+1}\right)$$

(8) $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & t \in [0,1] \\ t^2, & t \in [1,2] \\ e^{-t}, & t \in [2,3] \\ 0, & t > 3 \end{cases}$$

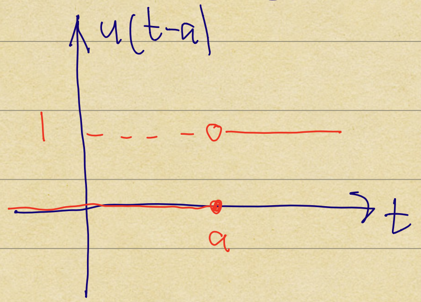
$$f(t)=\begin{cases}1, & t\in(0,1]\\ t^2, & t\in[1,2]\\ e^{-t}, & t\in(2,3]\\ 0, & t>3\end{cases} = \begin{cases}1, & t\in(0,1]\\ 0, & t\in(1,2]\\ 0, & t\in(2,3]\\ 0, & t>3\end{cases} +$$

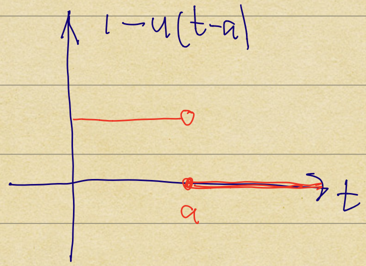
$$\begin{cases}0, & t\in(0,1]\\ t^2, & t\in(1,2]\\ 0, & t\in(2,3]\\ 0, & t>3\end{cases} + \begin{cases}0, & t\in(0,1]\\ 0, & t\in(1,2]\\ e^{-t}, & t\in(2,3]\\ 0, & t>3\end{cases}$$

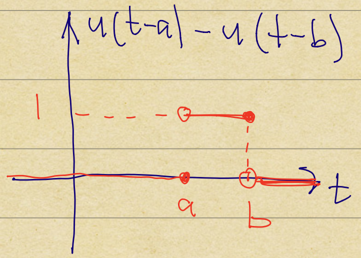
$$= \begin{cases}1, & t\in(0,1]\\ 0, & t>1\end{cases} + \begin{cases}0, & t\in(0,1]\\ t^2, & t\in(1,2]\\ 0, & t>2\end{cases} + \begin{cases}0, & t\in(0,2]\\ e^{-t}, & t\in(2,3]\\ 0, & t>3\end{cases}$$

$$= \begin{cases}1, & t\in(0,1]\\ 0, & t>1\end{cases} + t^2\begin{cases}0, & t\in(0,1]\\ 1, & t\in(1,2]\\ 0, & t>2\end{cases} + e^{-t}\begin{cases}0, & t\in(0,2]\\ 1, & t\in(2,3]\\ 0, & t>3\end{cases}$$

3 basic ingridients:







$\Rightarrow$

$$f(t) = \begin{cases} 1, & t \in (0,1] \\ 0, & t > 1 \end{cases} + t^2 \begin{cases} 0, & t \in (0,1] \\ 1, & t \in (1,2] \\ 0, & t > 2 \end{cases} + e^{-t} \begin{cases} 0, & t \in (0,2] \\ 1, & t \in (2,3] \\ 0, & t > 3 \end{cases}$$

$$= 1 - u(t-1) + t^2 (u(t-1) - u(t-2)) + e^{-t} (u(t-2) - u(t-3))$$

$$= 1 - u(t-1) + t^2 u(t-1) - t^2 u(t-2) + e^{-t} u(t-2) - e^{-t} u(t-3) = 1 + (t^2 - 1) u(t-1) + (e^{-t} - t^2) u(t-2) - e^{-t} u(t-3) \Rightarrow$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 + (t^2-1) u(t-1) + (e^{-t} - t^2) u(t-2) - e^{-t} u(t-3)\}$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{(((t-1)+1)^2 - 1) u(t-1)\} + \mathcal{L}\{(e^{-((t-2)+2)} - ((t-2)+2)^2) * u(t-2)\} - \mathcal{L}\{e^{-(t-3)-3} u(t-3)\} \overset{2TT}{=} \frac{1}{s} + \mathcal{L}\{(t+1)^2 - 1\} e^{-s}$$

$$+ \mathcal{L}\{e^{-t-2} - (t+2)^2\} e^{-2s} - \mathcal{L}\{e^{-t-3}\} e^{-3s}$$

$$= \frac{1}{s} + \mathcal{L}\{t^2 + 2t\} e^{-s} + \mathcal{L}\{e^{-2} e^{-t}\} e^{-2s} - \mathcal{L}\{t^2 + 4t + 4\} e^{-2s} - \mathcal{L}\{e^{-3} e^{-t}\} e^{-3s}$$

$$= \frac{1}{s} + (\mathcal{L}\{t^2\} + 2\mathcal{L}\{t\}) e^{-s} + e^{-2} e^{-2s} \mathcal{L}\{e^{-t}\} - (\mathcal{L}\{t^2\} + 4\mathcal{L}\{t\} + 4\mathcal{L}\{1\}) e^{-2s} - e^{-3} e^{-3s} \mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{s} + \left(\frac{2}{s^3} + \frac{2}{s^2}\right)e^{-s} + e^{-2s-2}\frac{1}{s+1}$$

$$- \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)e^{-2s} - e^{-3s-3}\frac{1}{s+1}$$