Ex.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s+1)(s^2+1)}\right\} = ?$$

Use partial fractions: (won't determine coeffs)

$$\frac{1}{(s^2-2s+1)(s^2+1)} = \frac{1}{(s-1)^2(s^2+1)} = \frac{A(s-1)+B}{(s-1)^2} + \frac{Cs+D}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{A(s-1)+B}{(s-1)^2}\right\} \overset{1\text{ST}}{=} e^t \mathcal{L}^{-1}\left\{\frac{As+B}{s^2}\right\}$$

$$= e^t\left(A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right) = e^t(A+Bt)$$

$$\mathcal{L}^{-1}\left\{\frac{Cs+D}{s^2+1}\right\} = C\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + D\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= C\cos t + D\sin t$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s+1)(s^2+1)}\right\} = Ae^t + Bte^t + C\cos t + D\sin t.$$

Ex. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+5s+6}\right\} \overset{2\text{ST}}{=} \mathcal{L}^{-1}\left\{\frac{1}{s^2+5s+6}\right\}(t-2)\,u(t-2)$

Now: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{A}{s+2} + \frac{B}{s+3}\right\} = Ae^{-2t} + Be^{-3t} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+5s+6}\right\} = \left[Ae^{-2(t-2)} + Be^{-3(t-2)}\right]u(t-2)$$

Ex.

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s^2+2s+2)(s+6)}\right\} \overset{2TT}{=} \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+2)(s+6)}\right\}(t-1)u(t-1)$$

We have $\frac{1}{(s^2+2s+2)(s+6)} = \frac{1}{((s+1)^2+1)(s+6)} = \frac{A(s+1)+B}{(s+1)^2+1} + \frac{C}{s+6}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+2)(s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{A(s+1)+B}{(s+1)^2+1}\right\} + C\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\}$$

$$\overset{1TT}{=} e^{-t}\mathcal{L}^{-1}\left\{\frac{As+B}{s^2+1}\right\} + C\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\}$$

$$= e^{-t}\left[A\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right] + C\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\}$$

$$= e^{-t}\left[A\cos t + B\sin t\right] + Ce^{-6t} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s^2+2s+2)(s+6)}\right\} \overset{2TT}{=} \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+2)(s+6)}\right\}(t-1)u(t-1)$$

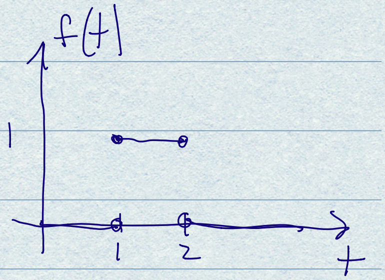
$$= e^{-(t-1)}\left[A\cos(t-1) + B\sin(t-1)\right] + Ce^{-6(t-1)}\Big]u(t-1)$$

Ex. Solve:

$$\begin{cases} y'' - 4y' + 5y = f(t), \\ y(0) = 0, \\ y'(0) = 0, \end{cases}$$

where

$$f(t) = \begin{cases} 0, & t<1 \\ 1, & t \in [1,2] \\ 0, & t>2 \end{cases}$$



Step I: Write $f(t)$ using step-function notation:

$$f(t) = u(t-1) - u(t-2)$$

Step II: Find the transform of $y(t)$:

$$\mathcal{L}\{y'' - 4y' + 5y\} = \mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-1) - u(t-2)\}$$

$$s^2 Y - 4sY + 5Y = \mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\} \overset{2TT}{=}$$

$$= e^{-s}\mathcal{L}\{1\} - e^{-2s}\mathcal{L}\{1\} = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$\Rightarrow Y(s) = \frac{e^{-s}}{s(s^2-4s+5)} - \frac{e^{-2s}}{s(s^2-4s+5)}$$

Step III: Find the inverse transform of $Y(s)$:

$$\frac{1}{s(s^2-4s+5)} = \frac{1}{s((s-2)^2+1)} = \frac{A}{s} + \frac{B(s-2)+C}{(s-2)^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s^2-4s+5)}\right\} = A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{B(s-2)+C}{(s-2)^2+1}\right\}$$

$$\overset{1TT}{=} A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + e^{2t}\mathcal{L}^{-1}\left\{\frac{Bs+C}{s^2+1}\right\}$$

$$= A + e^{2t}(B\cos t + C\sin t).$$ Therefore

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{e^{-s}}{s(s^2-4s+5)} - \frac{e^{-2s}}{s(s^2-4s+5)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{e^{-s}}{s(s^2-4s+5)} \right\} - \mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s(s^2-4s+5)} \right\} \quad \left\{ \begin{matrix} 2\pi \\ = \\ *2 \end{matrix} \right.$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s(s^2-4s+5)} \right\}(t-1)\, u(t-1) - \mathcal{L}^{-1}\left\{ \frac{1}{s(s^2-4s+5)} \right\}(t-2)\, u(t-2)$$

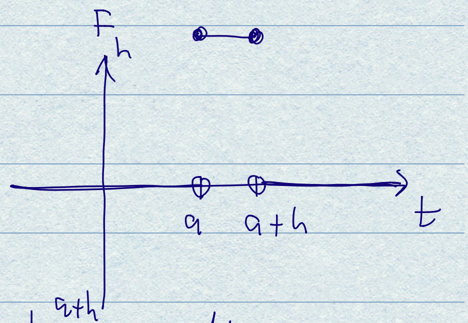
$$= \left(A + e^{2(t-1)}(B\cos(t-1) + C\sin(t-1)\right) u(t-1)$$

$$- \left(A + e^{2(t-2)}(B\cos(t-2) + C\sin(t-2)\right) u(t-2)$$

Delta-function:

Suppose we want to describe mathematically a large constant force pulse that has a very short duration:

$$F_h(t) = \begin{cases} 0, & t < a \\ \frac{1}{h}, & t \in [a, a+h] \\ 0, & t > a+h \end{cases}$$



$$\Rightarrow \int_0^\infty F_h(t)\, g(t) = \int_a^{a+h} \frac{1}{h} g(t)\, dt = \frac{1}{h}\int_a^{a+h} g(t)\, dt = \text{average}$$

value of g on the interval $[a, a+h]$

The pulse becomes stronger and more "concentrated" as $h \to 0$ $\Rightarrow$ formally,

$$\lim_{h\to 0} F_h(t) = \begin{cases} 0, & t<a \\ \infty, & t=a \\ 0, & t>a \end{cases}$$ – this is called a δ-"function", $\delta(t-a)$

Then, for any continuous function g, the following property holds:

$$\int_0^\infty \delta(t-a)g(t)\,dt \text{ "=" } \int_0^\infty \left(\lim_{h\to 0} F_h(t)\right)g(t)\,dt$$

$$= \lim_{h\to 0}\int_0^\infty F_h(t)g(t)\,dt = \lim_{h\to 0}\left(\text{average value of } g(t) \text{ on } [a,a+h]\right) = g(a)$$

Therefore, $\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & \text{otherwise} \end{cases}$, where

$$\int_0^\infty \delta(t-a)g(t)\,dt = g(a)$$

– infinite force applied at time a.

Now:

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty \delta(t-a)e^{-st}\,dt = e^{-sa}$$

Ex: Solve $\begin{cases} y''-y=\delta(t-1) \\ y(0)=y'(0)=0 \end{cases}$

$$\mathcal{L}\{y''-y\} = \mathcal{L}\{\delta(t-1)\}$$

$$s^2Y - Y = e^{-s}$$

$$(s^2-1)Y = e^{-s} \Rightarrow Y(s) = \frac{e^{-s}}{s^2-1}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2-1}\right\} \overset{2TT}{=}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}(t-1)\,u(t-1) = \sinh(t-1)\,u(t-1)$$

alterrative:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = \frac{1}{2}(e^{t}-e^{-t}).$$