Convolution:

Clearly, $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$

because the integral of a product is not equal to a product of integrals. Indeed, if this was true,

$$\frac{1}{s} = \mathcal{L}\{1\} = \mathcal{L}\{1\cdot 1\} = \mathcal{L}\{1\}\cdot\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

– nonsense! However, it turnes out that the product of transforms is meaningful.

Given two functions, $f(t)$ and $g(t)$, the convolution of f and g is defined as:

$$(f*g)(t) = \int_0^t f(p)g(t-p)\,dp$$ – a new function of t.

Examples: $1*1 = \int_0^t 1\cdot 1\,dp = t$

$$t*1 = \int_0^t p\cdot 1\,dp = \frac{p^2}{2}\Big|_0^t = \frac{t^2}{2}$$

$$t*t = \int_0^t p(t-p)\,dp = \int_0^t (pt-p^2)\,dp$$

$$= \left(t\frac{p^2}{2} - \frac{p^3}{3}\right)\Big|_0^t = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6}$$

$$t * e^{-t} = \int_0^t p e^{p-t}\,dp = e^{-t}\int_0^t p e^{p}\,dp \quad \left| \begin{array}{ll} u=p & du=dp \\ dv=e^p dp & v=e^p \end{array} \right|$$

$$= e^{-t}\left(pe^p\Big|_0^t - \int_0^t e^p\,dp\right) = e^{-t}\left(pe^p\Big|_0^t - e^p\Big|_0^t\right)$$

$$= e^{-t}\left(te^t - e^t + 1\right) = t - 1 + e^{-t}$$

For these examples: $\mathcal{L}\{1*1\} = \mathcal{L}\{t\} = \frac{1}{s^2}$, but

$\mathcal{L}\{1\} = \frac{1}{s} \Rightarrow \mathcal{L}\{1*1\} = \mathcal{L}\{1\}\mathcal{L}\{1\}$

$$\mathcal{L}\{t*1\} = \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2}\mathcal{L}\{t^2\}$$

$= \frac{1}{2}\frac{2!}{s^3} = \frac{1}{s^3}$, but $\mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{t\} = \frac{1}{s^2}$

$\Rightarrow \mathcal{L}\{t*1\} = \mathcal{L}\{t\}\mathcal{L}\{1\}$

$$\mathcal{L}\{t*e^{-t}\} = \mathcal{L}\{t-1+e^{-t}\}$$

$$= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} = \frac{s+1-s(s+1)+s^2}{s^2(s+1)} = \frac{\cancel{s}+1\cancel{-s^2}\cancel{-s}\cancel{+s^2}}{s^2(s+1)} = \frac{1}{s^2(s+1)}$$

$$= \frac{1}{s^2}\,\frac{1}{s+1} = \mathcal{L}\{t\}\,\mathcal{L}\{e^{-t}\}$$

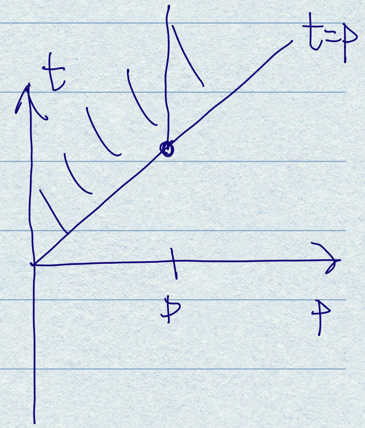
Conjecture: $\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$

Check:

$$\mathcal{L}\{(f*g)(t)\} = \int_0^\infty \left(\int_0^t f(p)g(t-p)\,dp\right) e^{-st}\,dt$$

$$= \int_0^\infty \int_0^t f(p)\, g(t-p)\, e^{-st}\, dp\, dt = \int_0^\infty \int_0^t f(p)\, g(t-p)\, e^{-s(t-p)-sp}\, dp\, dt$$

$$= \int_0^\infty \int_0^t f(p) e^{-sp}\, g(t-p)\, e^{-s(t-p)}\, dp\, dt$$



$$\underset{\substack{\text{reverse}\\ \text{order}\\ \text{of int.}}}{=} \int_0^\infty \int_p^\infty f(p) e^{-sp}\, g(t-p)\, e^{-s(t-p)}\, dt\, dp$$

$$\underset{d\tau = dt}{\overset{\tau = t-p}{=}} \int_0^\infty \int_0^\infty f(p) e^{-sp}\, g(\tau) e^{-s\tau}\, d\tau\, dp$$

$$= \int_0^\infty f(p) e^{-sp}\, dp \int_0^\infty g(\tau) e^{-s\tau}\, d\tau = \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\}$$

$$\Longleftrightarrow$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$$

Another property:

$$(g*f)(t) = \int_0^t g(p)\, f(t-p)\, dp \underset{\substack{p=t-u\\ du=-dp}}{\overset{u=t-p}{=}} \int_t^0 g(t-u)\, f(u)\, (-du)$$

$$= \int_0^t f(u)\, g(t-u)\, du = (f*g)(t)$$

Examples: Find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)s^2}\right\}$

Method I: $\frac{1}{(s^2+1)s^2} \overset{m=s^2}{=} \frac{1}{(m+1)m} = \frac{A}{m} + \frac{B}{m+1} = \frac{A(m+1)+Bm}{(m+1)m}$

$\Rightarrow A(m+1)+Bm = 1 \Rightarrow$ $m=0$: $A=1$

$m=-1$: $B=-1$

$\Rightarrow \frac{1}{(m+1)m} = \frac{1}{m} - \frac{1}{m+1} \Rightarrow \frac{1}{(s^2+1)s^2} = \frac{1}{s^2} - \frac{1}{s^2+1}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= t - \sin t$

Method II:

$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = (\sin t) * t$

$= \int_0^t (\sin p)(t-p)\,dp \overset{\text{parts}}{=} \left| \begin{matrix} u = t-p & du = -dp \\ dv = \sin p\,dp & v = -\cos p \end{matrix} \right|$

$= -(t-p)\cos p \Big|_0^t - \int_0^t \cos p\,dp = t - \sin p \Big|_0^t = t - \sin t$

Solve: $\begin{cases} y' + y = t * e^t \\ y(0) = 1 \end{cases}$

$$\Rightarrow \quad \mathcal{L}\{y'+y\} = \mathcal{L}\{t * e^t\} = \mathcal{L}\{t\}\,\mathcal{L}\{e^t\}$$

$$sY - 1 + Y = \frac{1}{s^2}\,\frac{1}{s-1}$$

$$(s+1)Y = 1 + \frac{1}{s^2}\,\frac{1}{s-1}$$

$$Y = \frac{1}{s+1} + \frac{1}{s^2(s-1)(s+1)}$$

$$= \frac{1}{s+1} + \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$+ C\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + D\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= e^{-t} + A + Bt + Ce^t + De^{-t}$$

$$= (D+1)e^{-t} + A + Bt + Ce^t$$

Now, determine the constants:

$$\frac{1}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$= \frac{As(s^2-1) + B(s^2-1) + Cs^2(s+1) + Ds^2(s-1)}{s^2(s-1)(s+1)}$$

$$\Rightarrow \quad As(s^2-1) + B(s^2-1) + Cs^2(s+1) + Ds^2(s-1) = 1$$

$s = 1: \quad 2C = 1 \Rightarrow C = \frac{1}{2}$

$s = -1: \quad -2D = 1 \Rightarrow D = -\frac{1}{2}$

$s = 0: \quad -B = 1 \Rightarrow B = -1$

$s = 2: \quad 6A + 3B + 12C + 4D = 1$

$6A - 3 + 6 - 2 = 1 \Rightarrow A = 0$

$$\Rightarrow \quad y(t) = \frac{1}{2}e^{-t} - t + \frac{1}{2}e^{t}$$