Ex 1: $y'' + 3y' + 2y = \cos x$

(1) Find the solution:

$$y'' + 3y' + 2y = 0$$

$\Rightarrow r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0$

$\Rightarrow r_1 = -1,\ r_2 = -2 \Rightarrow y_h = c_1 e^{-x} + c_2 e^{-2x}$

(2) Use $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$; $y_p'' = -A\cos x - B\sin x$

$$-A\cos x - B\sin x + 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = \cos x$$

$$(-A + 3B + 2A)\cos x + (-B - 3A + 2B)\sin x = \cos x$$

$$\begin{cases} A + 3B = 1 & *3 \\ B - 3A = 0 \end{cases} \Rightarrow \begin{array}{r} 3A + 9B = 3 \\ + \quad B - 3A = 0 \\ \hline 10B = 3 \end{array} \Rightarrow B = \frac{3}{10}$$

Also, $A = 1 - 3B = 1 - 3\cdot\frac{3}{10} = \frac{1}{10}$

$$\Rightarrow y_p(x) = \frac{1}{10}\cos x + \frac{3}{10}\sin x$$

$\Rightarrow \quad y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x}$
$\qquad + \frac{1}{10}\cos x + \frac{3}{10}\sin x$

Ex. 2 $\quad y'' - 2y' + 5y = e^x \cos 2x \qquad (\#)$

Start by solving $y'' - 2y' + 5y = 0$; characteristic equation $r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\Rightarrow \quad y(x) = e^x (c_1 \cos 2x + c_2 \sin 2x)$

To find a particular solution of $(\#)$

$$y(x) = Ae^x \cos 2x + Be^x \sin 2x$$

$\Rightarrow$ this $y(x)$ satisfies the homogeneous ODE $\Rightarrow$ cannot also solve $(\#)$. Consider instead

$$y(x) = Axe^x \cos 2x + Bxe^x \sin 2x$$

$\Rightarrow \quad y'(x) = Ae^x \cos 2x + \cancel{Axe^x \cos 2x} - \cancel{2Axe^x \sin 2x}$
$\quad + Be^x \sin 2x + \cancel{Bxe^x \sin 2x} + \cancel{2Bxe^x \cos 2x}$

$$= (A+2B)xe^x\cos 2x + (B-2A)xe^x\sin 2x$$
$$+ Ae^x\cos 2x + Be^x\sin 2x$$

$$y''(x) = (A+2B)e^x\cos x + (A+2B)xe^x\cos 2x$$
$$-2(A+2B)xe^x\sin 2x + (B-2A)e^x\sin 2x$$
$$+(B-2A)xe^x\sin 2x + 2(B-2A)xe^x\cos 2x$$
$$+Ae^x\cos 2x - 2Ae^x\sin 2x + Be^x\sin 2x + 2Be^x\cos 2x$$

substitute back into (#): $y_p'' - 2y_p' + 5y_p$

$$= (A+2B)e^x\cos 2x + (A+2B)xe^x\cos 2x$$
$$-2(A+2B)xe^x\sin 2x + (B-2A)e^x\sin 2x$$
$$+(B-2A)xe^x\sin 2x + 2(B-2A)xe^x\cos 2x$$
$$+Ae^x\cos 2x - 2Ae^x\sin 2x + Be^x\sin 2x + 2Be^x\cos 2x$$
$$-2\big((A+2B)xe^x\cos 2x + (B-2A)xe^x\sin 2x$$
$$+ Ae^x\cos 2x + Be^x\sin 2x\big)$$
$$+5\left(Axe^x\cos 2x + Bxe^x\sin 2x\right) = e^x\cos 2x$$

$$x e^x \cos 2x \Big(A + 2B + 2(B - 2A) - 2(A + 2B) + 5A\Big) + x e^x \sin 2x \Big(-2(A + 2B) + B - 2A - 2(B - 2A) + 5B\Big) + e^x \cos x \Big(A + 2B + A + 2B - 2A\Big) + e^x \sin x \Big(B - 2A - 2A + B - 2B\Big)$$

$$= e^x \cos 2x$$

$$\Rightarrow \quad 4B e^x \cos 2x - 4A e^x \sin x = e^x \cos 2x$$

$$\Rightarrow \begin{cases} 4B = 1 \\ 4A = 0 \end{cases} \Rightarrow \quad B = \frac{1}{4} \quad \Rightarrow \quad y_p = \frac{1}{4} x e^x \sin 2x$$

$\Rightarrow$

$$y(x) = e^x \left(c_1 \cos 2x + c_2 \sin 2x\right) + \frac{1}{4} x e^x \sin 2x$$

Ex. 3 $\quad y'' - 3y' + 2y = x^2 + e^x + x^3 e^{3x} + \cos 3x + x^2 \sin x$

Q: write the expression for y_p (do not determine the coefficients)

Step 1: Need to solve for y_h:

$$y'' - 3y' + 2y = 0 \Rightarrow r^2 - 3r + 2 = 0$$

$$\Rightarrow r = 1, 2 \Rightarrow y_h = c_1 e^x + c_2 e^{2x}$$

Step 2: $y_p = y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4} + y_{p_5}$

$y_{p_1} = Ax^2 + Bx + C$; $y_{p_2} = Dxe^x$ (since e^x solves homogeneous ODE)

$y_{p_3} = (Ex^3 + Fx^2 + Gx + H)e^{3x}$,

$y_{p_4} = I\cos 3x + J\sin 3x$,

$y_{p_5} = (Kx^2 + Lx + M)\sin x + (Nx^2 + Ox + P)\cos x$