The method of undetermined coefficients often does not work. Consider, for example, the equation

$$y'' + y = \sec x$$

If we guess that a particular solution looks like $y_p(x) = A\sec x$, then taking the derivative, we have

$$y_p' = A\sec x\tan x$$

which cannot be balanced by $\sec x$; if we try instead

$$y_p = A\sec x + B\sec x\tan x$$

this would not work either for the same reason: derivatives of this y_p cannot be balanced by functions out of which y_p is composed. A different method is needed.

The new method (called variation of parameters) works as follows.

Step 1: Solve the homogeneous equation:

$$y'' + y = 0 \implies r^2 + 1 = 0 \implies r = \sqrt{-1} = \pm i$$

$$\implies y_h(x) = c_1 \cos x + c_2 \sin x$$

Step 2: Replace c_1 and c_2 by unknown functions of x, $u_1(x)$ and $u_2(x)$ and set

$$y_p(x) = u_1(x)\cos x + u_2(x)\sin x$$

We want to determine u_1 and u_2 that make y_p a particular solution of

$$y'' + y = \sec x.$$

Compute: $y_p'(x) = u_1'\cos x - u_1 \sin x + u_2' \sin x$
$+ u_2 \cos x$

Now, because we are looking for one solution of the ODE and we have two unknown functions, we can assume that these functions satisfy any additional equation. We will set

$$u_1' \cos x + u_2' \sin x = 0,$$

then substituting this into (#)

$$y_p' = -u_1 \sin x + u_2 \cos x.$$

Now, take the second derivative:

$$y_p'' = -u_1' \sin x - u_1 \cos x + u_2' \cos x - u_2 \sin x$$

It follows that

$$y_p'' + y_p = -u_1' \sin x - \cancel{u_1 \cos x} + u_2' \cos x - \cancel{u_2 \sin x}$$
$$+ \cancel{u_1 \cos x} + \cancel{u_2 \sin x} = -u_1' \sin x + u_2' \cos x$$

and because $y_p'' + y_p = \sec x$, we get

$$-u_1' \sin x + u_2' \cos x = \sec x$$

Therefore u_1' and u_2' satisfy two equations

$$\begin{cases} u_1' \cos x + u_2' \sin x = 0, \\ -u_1' \sin x + u_2' \cos x = \sec x. \end{cases}$$

We solve this system for u_1' and u_2':

Multiplying the first equation by $\sin x$, the second by $\cos x$, and adding, we have

$$\sin x\left(\cancel{u_1'\cos x} + u_2'\sin x\right) + \cos x\left(\cancel{-u_1'\sin x} + u_2'\cos x\right)$$
$$= \cos x \cdot \sec x = 1$$

$$\Rightarrow (\sin^2 x + \cos^2 x)\,u_2' = 1 \Rightarrow u_2' = 1 \Rightarrow u_2 = x$$

Now, from the first equation:

$$u_1' = -u_2'\tan x = -\tan x$$

$$\Rightarrow u_1 = -\int \tan x\,dx = -\int \frac{\sin x}{\cos x}dx \underset{du=-\sin x\,dx}{\overset{u=\cos x}{=}}$$
$$= \int \frac{du}{u} = \ln|u| = \ln|\cos x|$$

Notice that I am not adding arbitrary constants to either of the integrals above because I am looking for a single particular solution. Therefore,

$$y_p(x) = u_1(x)\cos x + u_2(x)\sin x = \cos x\ln|\cos x| + x\sin x$$

In general, one does not need to remember this procedure, but know the formula that results from it:

Suppose the equation $a_2y''+a_1y'+a_0y=0$ has the general solution $y_h(x)=c_1y_1(x)+c_2y_2(x)$. and we want to find a particular solution of $a_2y''+a_1y'+a_0y=f(x)$. Then

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x), \text{ where}$$

$$u_1'=-\frac{y_2f}{W}, \quad u_2'=\frac{y_1f}{W}$$

and $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian.

Let's try to use this procedure on the example above:

$$y''+y=\sec x$$

Recall: $y_h=c_1\cos x+c_2\sin x$

$\Rightarrow$ $y_1(x)=\cos x$ and $y_2(x)=\sin x$

$$\Rightarrow \quad W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Because $f(x) = \sec x$:

$$u_1' = -\frac{y_2 f}{W} = -\frac{\sin x \cdot \sec x}{1} = -\tan x$$

$$u_2' = \frac{y_1 f}{W} = \frac{\cos x \sec x}{1} = 1$$

– the same equations as we saw above!

Ex. 1. $\quad y'' + y = \cos^2 x$

Step 1: Solve $y'' + y = 0 \Rightarrow$

$$r^2 + 1 = 0 \Rightarrow r = \sqrt{-1} = \pm i \Rightarrow$$

$$y_h = c_1 \underset{\substack{\shortparallel \\ y_1}}{\cos x} + c_2 \underset{\substack{\shortparallel \\ y_2}}{\sin x}$$

Step 2: Varion of parameters:

$$y_p = u_1(x) \cos x + u_2(x) \sin x \text{ , where}$$

$$u_1' = -\frac{f y_2}{W} \ , \quad u_2' = \frac{f y_1}{W}$$

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x$$

$$+ \sin^2 x = 1$$

$$\Rightarrow \quad u_1' = -\frac{\cos^2 x \sin x}{1} = -\cos^2 x \sin x$$

$$u_2' = \frac{\cos^2 x \cos x}{1} = \cos^3 x$$

$$\Rightarrow \quad u_1 = -\int \cos^2 x \sin x \, dx \underset{du=-\sin x dx}{\overset{u=\cos x}{=}} \int u^2 du$$

$$= \frac{u^3}{3} = \frac{\cos^3 x}{3}$$

$$u_2 = \int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$$

$$= \int (1-\sin^2 x) \cos x \, dx \underset{du=\cos x dx}{\overset{u=\sin x}{=}} \int (1-u^2) du$$

$$= u - \frac{u^3}{3} = \sin x - \frac{\sin^3 x}{3}$$

$$\Rightarrow \quad y_p = \frac{\cos^3 x}{3} \cos x + \left(\sin x - \frac{\sin^3 x}{3}\right) \sin x$$

$$\Rightarrow \quad y = C_1 \cos x + C_2 \sin x + \frac{\cos^4 x}{3} + \sin^2 x - \frac{\sin^4 x}{3}$$

Ex 2: $y'' - y = \frac{9x}{e^{3x}}$

Step 1: $y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$\Longrightarrow y_h(x) = c_1 e^{-x} + c_2 e^{x}$

Step 2: set $y_p = u_1 \underbrace{e^{-x}}_{y_1} + u_2 \underbrace{e^{x}}_{y_2}$

$$W(e^{-x}, e^{x}) = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = e^{-x}e^{x} + e^{x}e^{-x} = 2$$

$$u_1' = -\frac{y_2 f}{W} = -\frac{e^{x} \cdot \frac{9x}{e^{3x}}}{2} = -\frac{1}{2} e^{x} \cdot 9x e^{-3x}$$

$$\Rightarrow u_1' = -\frac{9}{2} x e^{-2x}$$

$$u_2' = \frac{y_1 f}{W} = \frac{e^{-x} \cdot 9x e^{-3x}}{2} = \frac{9}{2} x e^{-4x}$$

$$u_1 = \int \left(-\frac{9}{2} x e^{-2x}\right) dx = -\frac{9}{2} \int x e^{-2x} dx \quad \left| \begin{matrix} v = x & dv = dx \\ dw = e^{-2x} dx & w = -\frac{1}{2} e^{-2x} \end{matrix} \right|$$

$$= -\frac{9}{2}\left(-\frac{1}{2} x e^{-2x} + \frac{1}{2} \int e^{-2x} dx\right)$$

$$= -\frac{9}{2}\left(-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x}\right) = \frac{9}{4} e^{-2x}\left(x + \frac{1}{2}\right)$$

$$u_2 = \int \left(\frac{9}{2} x e^{-4x}\right) dx = \frac{9}{2} \int x e^{-4x} dx \quad \left| \begin{matrix} v = x & dv = dx \\ dw = e^{-4x} dx & w = -\frac{1}{4} e^{-4x} \end{matrix} \right|$$

$$= \frac{9}{2}\left(-\frac{1}{4}xe^{-4x} + \frac{1}{4}\int e^{-4x}dx\right)$$

$$= \frac{9}{2}\left(-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x}\right)$$

$$= -\frac{9}{8}e^{-4x}\left(x+\frac{1}{4}\right)$$

$$y_p = u_1 e^{-x} + u_2 e^{x} = \left(\frac{9}{4}e^{-2x}\left(x+\frac{1}{2}\right)\right)e^{-x}$$

$$-\frac{9}{8}e^{-4x}\left(x+\frac{1}{4}\right)e^{x}$$

$$= \frac{9}{4}e^{-3x}\left(x+\frac{1}{2}\right) - \frac{9}{8}e^{-3x}\left(x+\frac{1}{4}\right)$$

$$= \frac{9}{4}e^{-3x}\left(x+\frac{1}{2}-\frac{1}{2}\left(x+\frac{1}{4}\right)\right)$$

$$= \frac{9}{4}e^{-3x}\left(\frac{1}{2}x+\frac{3}{8}\right) = \frac{9}{8}e^{-3x}\left(x+\frac{3}{4}\right)$$

On the other hand,

$$y'' - y = \frac{9x}{e^{3x}} = 9xe^{-3x}$$

Undetermined coefficients:

$$y_p = (Ax+B)e^{-3x}$$

$$\Rightarrow \quad y_p' = Ae^{-3x} + (Ax+B)(-3)e^{-3x}$$

$$= (A - 3Ax - 3B)e^{-3x}$$

$$y_p'' = -3Ae^{-3x} + (A - 3Ax - 3B)(-3)e^{-3x}$$

$$= -3e^{-3x}(A + A - 3Ax - 3B)$$

$$\Rightarrow y_p'' - y_p = -3e^{-3x}(2A - 3Ax - 3B)$$

$$- e^{-3x}(Ax + B) = 9xe^{-3x}$$

$$-3(2A - 3Ax - 3B) - (Ax + B) = 9x$$

$$-6A + 9Ax + 9B - Ax - B = 9x$$

$$\Rightarrow \quad 8Ax + 8B - 6A = 9x \quad \text{for all } x$$

$$\Rightarrow \quad 8A = 9 \Rightarrow A = \frac{9}{8}$$

$$8B - 6A = 0 \Rightarrow 4B - 3A = 0 \Rightarrow B = \frac{3}{4}A$$

$$\Rightarrow \quad B = \frac{3}{4} \cdot \frac{9}{8} = \frac{27}{32}$$

$$y_p = (Ax + B)e^{-3x} = \left(\frac{9}{8}x + \frac{27}{32}\right)e^{-3x}$$

$$= \frac{9}{8}\left(x + \frac{3}{4}\right)e^{-3x}$$