Pb 1. $y = c_1 e^{2x} + c_2 e^{-x}$

$$y' = 2c_1 e^{2x} - c_2 e^{-x}$$

$$y'' = 4c_1 e^{2x} + c_2 e^{-x}$$

$$y'' - y' - 2y = (4c_1 e^{2x} + c_2 e^{-x}) - (2c_1 e^{2x} - c_2 e^{-x})$$

$$-2(c_1 e^{2x} + c_2 e^{-x})$$

$$= (4c_1 - 2c_1 - 2c_1 e^{2x}) + (c_2 + c_2 - 2c_2) e^{-x} = 0$$

$\Rightarrow$ y is a solution of the equation

The equation is second order, linear.

Pb 2. (a) $f(x,y) = (x^2+y^2)^{1/2}$

$$\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{1/2}}$$ - not continuous at the origin.

$\Rightarrow$ IVP has a unique solution as long as the initial point is not at the origin.

(b) $f(x,y) = \frac{x-y}{x+y}$ - solution is unique as long as the initial point is not on $y = -x$.

Pb 3. (a) $\frac{dy}{dx} = y/x^2$

$$\int \frac{dy}{y} = \int \frac{dx}{x^2} \Rightarrow \ln|y| = -\frac{1}{x} + C$$

(b) $y' + y^2 \sin x = 0$

$$\frac{dy}{dx} = -y^2 \sin x \Rightarrow \frac{dy}{y^2} = -\sin x \, dx$$

$$\int \frac{dy}{y^2} = -\int \sin x \, dx \Rightarrow -\frac{1}{y} = \cos x + C$$

(c) $y' = 1 + x + y^2 + xy^2 = (1+x)(1+y^2)$

$$\frac{dy}{1+y^2} = (1+x)dx \Rightarrow \int \frac{dy}{1+y^2} = \int (1+x) dx$$

$$\operatorname{atan} y = x + \frac{x^2}{2} + C$$

(d) $(1+y^2) y' = x^2 \Rightarrow (1+y^2) dy = x^2 dx$

$$\Rightarrow \int (1+y^2) dy = \int x^2 dx \Rightarrow y + \frac{y^3}{3} = \frac{x^3}{3} + C$$

Pb4. (a) $x^2y' + 3xy = \frac{\sin x}{x}$, $x>0$

$y' + \frac{3}{x}y = \frac{\sin x}{x^3} \Rightarrow \mu(x) = e^{\int \frac{3}{x}dx}$
$= e^{3\ln x} = x^3 \Rightarrow$

$(x^3 y)' = \sin x \Rightarrow x^3 y = \int \sin x\, dx = -\cos x + C$

(b) $y' + \frac{2y}{x} = \frac{e^x}{x}$, $x>0$

$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$\Rightarrow (x^2y)' = xe^x \Rightarrow x^2y = \int xe^x dx = \left| \begin{array}{ll} u=x & du=dx \\ dv=e^x dx & v=e^x \end{array} \right|$

$= xe^x - \int e^x dx = xe^x - e^x + C$

$\Rightarrow x^2 y = xe^x - e^x + C$

(c) $y' + 3y = x + e^{-2x}$; $\mu(x) = e^{\int 3dx} = e^{3x}$

$(e^{3x}y)' = xe^{3x} + e^x$

$e^{3x}y = \int xe^{3x}dx + \int e^x dx$;

$\int xe^{3x}dx = \left| \begin{array}{ll} u=x & du=dx \\ dv=e^{3x}dx & v=\frac{1}{3}e^{3x} \end{array} \right| = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}dx$

$$= \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C \Rightarrow$$

$$e^{3x}y = e^{x} + \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$$

(d) $y' - 2y = x^2 e^{2x}$; $\mu(x) = e^{-\int 2dx} = e^{-2x}$

$$\Rightarrow (e^{-2x}y)' = x^2 \Rightarrow$$

$$e^{-2x}y = \frac{x^3}{3} + C$$

<u>Pb5:</u> (a) $y' = \frac{y-x}{y+x}$

$y = vx \Rightarrow \frac{y-x}{y+x} = \frac{vx-x}{vx+x} = \frac{v-1}{v+1}$ - homogeneous

Complete the substitution: $y' = v'x + v$

$$\Rightarrow x\frac{dv}{dx} + v = \frac{v-1}{v+1}$$

$$x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1} = -\frac{v^2+1}{v+1}$$

$$\frac{v+1}{v^2+1}dv = -\frac{dx}{x}$$

$$\int \frac{v+1}{v^2+1}\,dv = \frac{1}{2}\int \frac{2v\,dv}{v^2+1} + \int \frac{dv}{v^2+1} = \frac{1}{2}\ln(v^2+1) + \tan^{-1} v$$

$$-\int \frac{dx}{x} = -\ln|x| + C \Rightarrow$$

$$\frac{1}{2}\ln(v^2+1) + \tan^{-1} v = -\ln|x| + C \Rightarrow$$

$$\frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2+1\right) + \tan^{-1}\frac{y}{x} = -\ln|x| + C.$$

(b) $(y^2+yx)\,dx + x^2\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+yx}{x^2}$$

$y = vx$: $\quad -\dfrac{y^2+yx}{x^2} = -\dfrac{v^2x^2+vx^2}{x^2} = -(v^2+v)$

— homogeneous;

$$\Rightarrow \quad xv' + v = -v^2 - v$$

$$x\frac{dv}{dx} = -v^2 - 2v = -(v^2+2v) = -v(v+2)$$

$$\frac{dv}{v(v+2)} = -\frac{dx}{x} \quad \text{Partial fractions:}$$

$$\left(\frac{1}{2}\frac{1}{v} - \frac{1}{2}\frac{1}{v+2}\right)dv = -\frac{dx}{x}; \quad \text{integrate:}$$

$$\frac{1}{2}\ln|v| - \frac{1}{2}\ln|v+2| = -\ln|x| + C \Rightarrow$$

$$\frac{1}{2}\ln\left|\frac{y}{x}\right| - \frac{1}{2}\ln\left|\frac{y}{x}+2\right| = -\ln|x| + C$$

(c) $xy' - y = \sqrt{x^2+y^2}$: $y = vx$:

$$xy' - vx = \sqrt{x^2+v^2x^2} \Rightarrow$$

$$y' = v + \frac{1}{x}\sqrt{x^2+v^2x^2} = v + \sqrt{1+v^2} \text{ - homogeneous}$$

$$xv' + \cancel{v} = \cancel{v} + \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} = \ln|x| + C$$

Trig substitution: $\displaystyle\int \frac{dv}{(v^2+1)^{1/2}} \underset{dv = \cosh\theta\, d\theta}{\overset{v=\sinh\theta}{=}}$

$$= \int \frac{\cosh\theta\, d\theta}{(1+\sinh^2\theta)^{1/2}} = \int \frac{\cancel{\cosh\theta}\, d\theta}{\cancel{\cosh\theta}} = \theta + C$$

$$= \sinh^{-1}(v) \Rightarrow$$

$$\sinh^{-1} v = \ln|x| + C$$

$$\Rightarrow \sinh^{-1}\frac{y}{x} = \ln|x| + C$$

(d) $x^2y' + 2xy = y^3$ - Bernoulli's eq.

$v = y^{1-3} = y^{-2} \Rightarrow y = v^{-1/2}$

$y' = -\frac{1}{2} v^{-3/2} v'$

$-\frac{x^2}{2} v^{-3/2} v' + 2xv^{-1/2} = (v^{-1/2})^3 = v^{-3/2}$

$\Rightarrow -\frac{x^2}{2} v' + 2xv = 1$

$v' - \frac{4}{x} v = -\frac{2}{x^2} \Rightarrow \mu(x) = e^{-4\int \frac{dx}{x}}$

$= e^{-4\ln x} = \frac{1}{x^4}$

$\left(\frac{1}{x^4} v\right)' = -\frac{2}{x^2} \cdot \frac{1}{x^4} = -\frac{2}{x^6}$

$\frac{v}{x^4} = -2\int \frac{dx}{x^6} = \frac{2}{5} x^{-5} + c \Rightarrow v = \frac{2}{5x} + cx^4$

$\Rightarrow y = v^{-1/2} = \left(\frac{2}{5x} + cx^4\right)^{-1/2}$

Pb6: (a) $y' + 2y = xe^{-2x}$, $y(1) = 0$

$\mu(x) = e^{\int 2dx} = e^{2x} \Rightarrow (e^{2x}y)' = xe^{-2x} \cdot e^{2x} = x$

$e^{2x}y = \int x = \frac{x^2}{2} + c$

$$e^{2\cdot 1}\cdot 0 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$$

$$e^{2x}y = \frac{x^2}{2} - \frac{1}{2} \quad //$$

(b) $y' = \dfrac{2x}{y + x^2 y}$, $y(0) = -2$

$$\frac{dy}{dx} = \frac{2x}{y(1+x^2)} \Rightarrow \int y\,dy = \int \frac{2x\,dx}{x^2+1}$$

$$\frac{y^2}{2} = \ln(x^2+1) + C$$

$$\frac{(-2)^2}{2} = \ln(0+1) + C \Rightarrow 2 = C$$

$$\frac{y^2}{2} = \ln(x^2+1) + 2 \quad //$$

(c) $\sin 2x\,dx + \cos 3y\,dy = 0$, $y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$

$$\int \cos 3y\,dy = -\int \sin 2x\,dx$$

$$\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + C$$

$$\frac{1}{3}\sin 3\left(\frac{\pi}{3}\right) = \frac{1}{2}\cos 2\frac{\pi}{2} + C$$

$$\frac{1}{3}\sin \pi = \frac{1}{2}\cos \pi + C$$

$$0 = -\frac{1}{2} + C \Rightarrow c = \frac{1}{2}$$

$$\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + \frac{1}{2} \quad //$$

Pb 10:

$$m(t) = m(0)e^{-kt}$$

$$m(0) = 50; \quad m(3) = 10 \Rightarrow$$

$$10 = m(3) = 50e^{-k \cdot 3} = 50(e^{-k})^3$$

$$\Rightarrow (e^{-k})^3 = \frac{1}{5} \Rightarrow e^{-k} = \left(\frac{1}{5}\right)^{1/3}$$

$$\Rightarrow m(t) = 50(e^{-k})^t = 50\left(\frac{1}{5}\right)^{t/3}$$

$$\Rightarrow \frac{m(4)}{m(0)} = \frac{m(4)}{50} = \left(\frac{1}{5}\right)^{4/3} \Rightarrow$$

Answer: $\left(\frac{1}{5}\right)^{4/3} \cdot 100$

Pb 11: (a) $(2x+3)dx + (2y-2)dy = 0$

$$\frac{\partial}{\partial y}(2x+3) = 0; \frac{\partial}{\partial x}(2y-2) = 0$$

$\Rightarrow$ exact.

$$\frac{\partial f}{\partial x} = 2x+3; \frac{\partial f}{\partial y} = 2y-2$$

$$\Rightarrow f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int (2x+3)\,dx = x^2+3x+C(y)$$

$$2y-2 = \frac{\partial f}{\partial y} = C'(y) \Rightarrow C(y) = \int (2y-2)\,dy$$
$$= y^2-2y$$

$\Rightarrow f(x,y) = x^2+3x+y^2-2y \Rightarrow$ Solution:

$$x^2+3x+y^2-2y = C$$

(b) $(2xy^2+2y)\,dx + (2x^2y+2x)\,dy = 0$

$$\frac{\partial}{\partial y}(2xy^2+2y) = 4xy+2$$

– exact

$$\frac{\partial}{\partial x}(2x^2y+2x) = 4xy+2$$

$$\frac{\partial f}{\partial x} = 2xy^2+2y, \quad \frac{\partial f}{\partial y} = 2x^2y+2x$$

$$\Rightarrow f(x,y) = \int (2xy^2+2y)\,dx = x^2y^2+2xy+C(y)$$

$$\cancel{2x^2y} + \cancel{2x} = \frac{\partial f}{\partial y} = \cancel{2x^2y} + \cancel{2x} + C'(y) \Rightarrow C(y) = 0$$

$\Rightarrow f(x,y) = x^2y^2+2xy \Rightarrow$ Sol-n: $x^2y^2+2xy = C$

(c) $(x\ln y + xy)dx + (y\ln x + xy)dy = 0;\ x,y>0$

$$\frac{\partial}{\partial y}(x\ln y + xy) = \frac{x}{y} + x$$

$$\frac{\partial}{\partial x}(y\ln x + xy) = \frac{y}{x} + y$$

not equal $\Downarrow$ not exact.

(d) $(ax+by)dx + (bx+cy)dy = 0$

$$\frac{\partial}{\partial y}(ax+by) = b\ ;\ \frac{\partial}{\partial x}(bx+cy) = b$$

$\Rightarrow$ exact

$$\frac{\partial f}{\partial x} = ax+by\ ;\ \frac{\partial f}{\partial y} = bx+cy$$

$$f(x,y) = \int (bx+cy)\,dy = bxy + \frac{cy^2}{2} + Q(x)$$

$$ax+\cancel{by} = \frac{\partial f}{\partial x} = \cancel{b}y + Q'(x)$$

$$\Rightarrow Q'(x) = ax \Rightarrow Q(x) = \frac{ax^2}{2}$$

$$f(x,y) = bxy + \frac{cy^2}{2} + \frac{ax^2}{2} \Rightarrow$$

solution: $bxy + \frac{cy^2}{2} + \frac{ax^2}{2} = k.$

Pb 12:

$$y_1 = e^{2x} \qquad y_2 = e^{-x} \qquad y_3 = e^{x}$$
$$y_1' = 2e^{2x} \qquad y_2' = -e^{-x} \qquad y_3' = e^{x}$$
$$y_1'' = 4e^{2x} \qquad y_2'' = e^{-x} \qquad y_3'' = e^{x}$$
$$y_1''' = 8e^{2x} \qquad y_2''' = -e^{-x} \qquad y_3''' = e^{x}$$

$$y_1''' - 2y_1'' - y_1' + 2y_1 = 8e^{2x} - 2\cdot 4e^{2x} - 2e^{2x} + 2e^{2x} = 0$$
$$y_2''' - 2y_2'' - y_2' + 2y_2 = -e^{-x} - 2e^{-x} + e^{-x} + 2e^{-x} = 0$$
$$y_3''' - 2y_3'' - y_3' + 2y_3 = e^{x} - 2e^{x} - e^{x} + 2e^{x} = 0$$

— all are solutions

$$W(e^{2x}, e^{-x}, e^{x}) = \begin{vmatrix} e^{2x} & e^{-x} & e^{x} \\ 2e^{2x} & -e^{-x} & e^{x} \\ 4e^{2x} & e^{-x} & e^{x} \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} -e^{-x} & e^{x} \\ e^{-x} & e^{x} \end{vmatrix} - e^{-x}\begin{vmatrix} 2e^{2x} & e^{x} \\ 4e^{2x} & e^{x} \end{vmatrix} + e^{x}\begin{vmatrix} 2e^{2x} & -e^{-x} \\ 4e^{2x} & e^{-x} \end{vmatrix}$$

$$= e^{2x}(-1-1) - e^{-x}(2e^{3x} - 4e^{3x})$$
$$+ e^{x}(2e^{x} + 4e^{x}) = \cancel{-2e^{2x}} \cancel{+2e^{2x}} + 6e^{2x}$$

$= 6e^{2x} \neq 0 \Rightarrow e^{2x}, e^{-x}, e^{x}$ are linearly independent $\Rightarrow$ the form fund. soln.

Pb 13: $f_1 = x^2$, $f_2 = 2x^2 - 3x$, $f_3 = x$, $f_4 = 1$

$$\Rightarrow f_2(x) = 2f_1(x) + 0 f_4(x) - 3 f_3(x)$$

$\Rightarrow$ linearly dependent.

Pb 14: $x^2 y'' + 2xy' = 0$

$$y'' + \frac{2}{x} y' = 0 \Rightarrow$$

$$y_2(x) = 1 \cdot \int \frac{e^{-\int \frac{2}{x} dx}}{1^2} dx$$

$$= \int e^{-2\ln x} dx = \int e^{\ln x^{-2}} dx$$

$$= \int x^{-2} dx = -\frac{1}{x}; \quad x > 0$$

Pb 15: (a)

$$y_2 = e^{6x} \int \frac{e^{-\int (-4) dx}}{(e^{6x})^2} dx$$

$$= e^{6x} \int \frac{e^{4x}}{e^{12x}} dx = e^{6x} \int e^{-8x} dx$$

$$= e^{6x}\left(-\frac{1}{8}\right)e^{-8x} = -\frac{1}{8}e^{-2x}.$$

(b)

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0$$

$$y_2(x) = x^{-1}\int \frac{e^{-\int \frac{3}{x}dx}}{(x^{-1})^2} = x^{-1}\int \frac{x^{-3}}{x^{-2}}dx$$

$$= \frac{1}{x}\ln x.$$