Problem 1:

$$y'' - 3y' - 4y = 8x + 2\sin x + 5e^{-x}$$

(1) $r^2 - 3r - 4 = 0 \Rightarrow r = -1, 4$

$$y_h = C_1 e^{-x} + C_2 e^{4x}$$

(2) $y_{p_1} = Ax + B \rightarrow y'_{p_1} = A,\ y''_{p_1} = 0$

$$\Rightarrow 0 - 3A - 4(Ax+B) = 8x$$

$$-4A = 8,\quad -3A - 4B = 0$$

$$\Rightarrow A = -2,\quad B = \frac{3}{2}$$

$$y_{p_1} = -2x + \frac{3}{2}$$

(3) $y_{p_2} = C\sin x + D\cos x$

$$\Rightarrow y'_{p_2} = C\cos x - D\sin x;\ y''_{p_2} = -C\sin x - D\cos x$$

$$-C\sin x - D\cos x - 3(C\cos x - D\sin x)$$
$$-4(C\sin x + D\cos x) = 2\sin x$$

$$\Rightarrow -C + 3D - 4C = 2$$

$$-D-3C-4D=0 \Rightarrow C=-\frac{5D}{3}$$

and

$$3D+\frac{25D}{3}=2 \Rightarrow D=\frac{3}{17}$$

$$\Rightarrow C=-\frac{5}{17}$$

$$y_{P_2}=-\frac{5}{17}\sin x+\frac{3}{17}\cos x$$

(3) $y_{P_3}=Axe^{-x} \Rightarrow y'_{P_3}=Ae^{-x}-xe^{-x}=A(1-x)e^{-x}$

$$y''_{P_3}=-Ae^{-x}-A(1-x)e^{-x}=A(x-2)e^{-x}$$

$$\Rightarrow A(x-2)e^{-x}-3A(1-x)e^{-x}-4Axe^{-x}=5e^{-x}$$

$$-2A-3A=5 \Rightarrow A=-1$$

$$y_{P_3}=-xe^{-x}$$

$$\Rightarrow y(x)=C_1e^{-x}+C_2e^{4x}+2x+\frac{3}{2}$$
$$-\frac{5}{17}\sin x+\frac{3}{17}\cos x-xe^{-x}$$

Problem 2:

$$y''+y=\frac{1}{\cos x}, \quad 0<x<\frac{\pi}{2}$$

$$r^2+1=0 \Rightarrow r=\pm i$$

$$y_h = c_1 \cos x + c_2 \sin x \,;$$

$y_p = u(x)\cos x + v(x)\sin x$, where

$$u' = -\frac{\frac{1}{\cos x}\sin x}{W}$$

$$v' = \frac{\frac{1}{\cos x}\cos x}{W}, \quad \text{and}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\Rightarrow \quad u' = -\frac{\sin x}{\cos x}, \quad v' = 1$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$u = -\int \frac{\sin x}{\cos x} \overset{p=\cos x}{\underset{dp=-\sin x\,dx}{=}} \qquad v = x$$

$$= \int \frac{dp}{p} = \ln|p| = \ln|\cos x|$$

$$\Rightarrow y_p(x) = \cos x \ln|\cos x| + x \sin x$$

Problem 3.

(a) $y'' - 2y' + 2y = 0$

$$r^2 - 2r + 2 = 0 \Rightarrow r = \frac{2 \pm (4-8)^{1/2}}{2}$$

$$r = 1 \pm i$$

$$y(x) = e^x (c_1 \cos x + c_2 \sin x)$$

(b) $9y'' - 6y' + y = 0$

$$9r^2 - 6r + 1 = 0$$

$$r = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{1}{3}$$

$$y = c_1 e^{x/3} + c_2 x e^{x/3}$$

(c) $y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$$y = c_1 e^{-x} + c_2 e^x$$

(d) $y''' + y = 0 \Rightarrow r^3 + 1 = 0$

$$\Rightarrow (r+1)(r^2 - r + 1)$$

$$r = -1 \quad \text{or} \quad r = \frac{1 \pm (1-4)^{1/2}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$$

Problem 4.

(1) $y'' + 8y' - 9y = 0$

$$r^2 + 8r - 9 = 0$$

$$r = -8, 1$$

$$y = c_1 e^{-8x} + c_2 e^{x}$$

$$y' = -8c_1 e^{-8x} + c_2 e^{x}$$

$$1 = y(1) = c_1 e^{-8} + c_2 e$$

$$0 = y'(1) = -8c_1 e^{-8} + c_2 e$$

$$1 = 9c_1 e^{-8} \Rightarrow c_1 = \frac{e^8}{9}$$

$$c_2 = \frac{1}{e} - c_1 e^{-9} = \frac{1}{e} - \frac{1}{9e} = \frac{8}{9e}$$

$$y = \frac{e^8}{9} e^{-8x} + \frac{8}{9e} e^{x}$$

(2) $y''+y=0 \Rightarrow r^2+1=0\,,\; r=\pm i$

$$y(x)=c_1\cos x+c_2\sin x$$

$$y'(x)=-c_1\sin x+c_2\cos x$$

$$2=y\left(\tfrac{\pi}{3}\right)=c_1\frac{1}{2}+c_2\frac{\sqrt{3}}{2}$$

$$-4=y'\left(\tfrac{\pi}{3}\right)=-c_1\frac{\sqrt{3}}{2}+c_2\frac{1}{2}$$

$$\Rightarrow\quad c_1+\sqrt{3}\,c_2=4 \qquad \sqrt{3}\,c_1+3c_2=4\sqrt{3}$$

$$-\sqrt{3}\,c_1+c_2=-8 \qquad -\sqrt{3}\,c_1+c_2=-8$$

$$4c_2=4\sqrt{3}-8$$

$$\Rightarrow c_2=\sqrt{3}-2$$

$$c_1=4-\sqrt{3}\,c_2=4-\sqrt{3}(\sqrt{3}-2)$$

$$=1+2\sqrt{3}$$

$$y(x)=(1+2\sqrt{3})\cos x+(\sqrt{3}-2)\sin x$$

(3) $y''+y=0 \Rightarrow y=c_1\cos x+c_2\sin x$

$$y'=-c_1\sin x+c_2\cos x$$

$$0 = y'(0) = c_2$$
$$2 = y'\left(\frac{\pi}{2}\right) = -c_1 \quad \Rightarrow \quad y = -2\cos x$$
$$c_1 = -2$$

Problem 5. $\quad y'' + 2y' + y = 0$

$$y_h = c_1 e^{-x} + c_2 x e^{-x}$$

(a) $\quad y_{p_1} = Ax^2 + Bx + C \Rightarrow y_{p_1}' = 2Ax + B$

$$y_{p_1}'' = 2A$$

$$\Rightarrow 2A + 2(2Ax + B) + Ax^2 + Bx + C = x^2$$

$$\Rightarrow A = 1; \quad 4A + B = 0 \Rightarrow B = -4$$
$$2A + 2B + C = 0 \Rightarrow C = 6$$

$$y_{p_1}(x) = x^2 - 4x + 6$$

(b) $\quad y_{p_2} = u(x)e^{-x} + v(x)xe^{-x}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}$$

$$u' = -\frac{\cancel{e^{-x}} \ln x \, x \cancel{e^{-x}}}{\cancel{e^{-2x}}} = -x \ln x$$

$$u = -\int x \ln x \, dx \quad \left| \begin{matrix} w = \ln x & dw = \frac{1}{x} dx \\ dz = x\,dx & z = \frac{x^2}{2} \end{matrix} \right|$$

$$= -\frac{x^2}{2} \ln x + \int \frac{x^2}{2} \frac{1}{x} dx = -\frac{x^2}{2} \ln x + \frac{x^2}{4}$$

$$v' = \frac{\cancel{e^{-x}} \ln x \cancel{e^{-x}}}{\cancel{e^{-2x}}}$$

$$v = \int \ln x \, dx = \left| \begin{matrix} w = \ln x & dw = \frac{dx}{x} \\ dz = dx & z = x \end{matrix} \right|$$

$$= x \ln x - \int dx = x(\ln x - 1)$$

$$y_{P_2} = \frac{x^2}{4}(1 - 2\ln x) e^{-x} + x^2 (\ln x - 1) e^{-x}$$

$$= \frac{x^2}{4}(1 - 2\ln x + 4 \ln x - 4) e^{-x}$$

$$= \frac{x^2}{4}(2 \ln x - 3) e^{-x}$$

$$\Rightarrow y(x) = c_1 e^{-x} + c_2 x e^{-x} + x^2 - 4x + 6 + \frac{x^2}{4}(2\ln x - 3) e^{-x}$$

Pb 6:

$$F(s) = \mathcal{L}\{e^{-t}\sin t\} = \int_0^\infty e^{-st} e^{-t} \sin t \, dt$$

$$= \int_0^\infty e^{-(s+1)t} \sin t \, dt;$$

$$\Rightarrow F(s) = \int_0^\infty e^{-(s+1)t} \sin t \, dt$$

$$= -e^{-(s+1)t}\cos t \Big|_0^\infty - (s+1)\int_0^\infty e^{-(s+1)t}\cos t \, dt$$

$$\overset{s+1>0}{=} 1 - (s+1)\int_0^\infty e^{-(s+1)t}\cos t \, dt$$

$$= \left| \begin{array}{ll} u = e^{-(s+1)t} & du = -(s+1)e^{-(s+1)t}\,dt \\ dv = \cos t \, dt & V = \sin t \end{array} \right|$$

$$= 1 - (s+1)\Big[\underbrace{e^{-(s+1)t}\sin t \Big|_0^\infty}_{0} + (s+1)\int_0^\infty e^{-(s+1)t}\sin t \, dt\Big]$$

$$= 1 - (s+1)^2 \int_0^\infty e^{-(s+1)t}\sin t \, dt = 1 - (s+1)^2 F(s)$$

$$\Rightarrow F(s) = 1 - (s+1)^2 F(s) \Rightarrow$$

$$F(s)\left(1 + (s+1)^2\right) = 1 \Rightarrow F(s) = \frac{1}{(s+1)^2 + 1}$$

is $s+1>0$.

Pb 7: (1) $mg = kl \iff$

$$1 \cdot 9{,}8 = k \cdot 9{.}8 \cdot 10^{-2}$$

$$\Rightarrow k = 100 \text{ N/m}.$$

(2) $1 \cdot \ddot{x} = -100x \Rightarrow \ddot{x} + 100x = 0,$

also, $x(0) = -1$, $\dot{x}(0) = 0 \Rightarrow$

$r^2 + 100 = 0 \Rightarrow r = \pm 10i$

$x(t) = c_1 \cos 10t + c_2 \sin 10t,$

$\dot{x}(t) = -10c_1 \sin 10t + 10c_2 \cos 10t$

$$\left.\begin{array}{l} -1 = x(0) = c_1 \\ 0 = \dot{x}(0) = 10c_2 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = -1 \\ c_2 = 0 \end{array} \Rightarrow$$

$x(t) = -\cos 10t \Rightarrow$

Amplitude $= 1$ m,

Period $= \frac{2\pi}{10} = \frac{\pi}{5}$ s,

frequency $= \frac{5}{\pi}$ s^{-1}.

Pb 8. (1) $mg = kl \Leftrightarrow$

$$0.5 \cdot 9.8 = k \cdot 0.098$$

$\Rightarrow k = 50 \text{ N/m}$

(2) $0.5\ddot{x} = -50x - 6\dot{x} \Rightarrow$

$$\begin{cases} \ddot{x} + 12\dot{x} + 100x = 0 \\ x(0) = 0 \\ \dot{x}(0) = 8 \end{cases}$$

$r^2 + 12r + 100 = 0 \Rightarrow r = \dfrac{-12 \pm (144 - 400)^{1/2}}{2}$

$= \dfrac{-12 \pm 16i}{2} = -6 \pm 8i$

$x(t) = e^{-6t}(\cancel{c_1 \cos 8t} + c_2 \sin 8t)$

$0 = x(0) = c_1 \Rightarrow c_1 = 0$

$\dot{x}(t) = c_2(-6e^{-6t}\sin 8t + e^{-6t} \cdot 8\cos 8t)$

$8 = \dot{x}(0) = 8c_2 \Rightarrow c_2 = 1$

$\Rightarrow x(t) = e^{-6t}\sin 8t$

Problem 9. Find the Laplace transform of a given function

(a). $f(t) = 5 - e^{2t} + 6t^2$, **(b).** $g(t) = t^4 e^{5t} - e^t \cos\sqrt{7}t$,
(c). $p(t) = (t+1)^2$, **(d.)** $h(t) = \sin(t + 2\pi)$.

(a) $\mathcal{L}\{5 - e^{2t} + 6t^2\} = \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}$

(b) $\mathcal{L}\{t^4 e^{5t} - e^t \cos\sqrt{7}t\} = \mathcal{L}\{t^4 e^{5t}\} - \mathcal{L}\{e^t \cos\sqrt{7}t\}$

$= \mathcal{L}\{t^4\}(s-5) - \mathcal{L}\{\cos\sqrt{7}t\}(s-1) = \frac{4!}{(s-5)^5} - \frac{s-1}{(s-1)^2+7}$

(c) $\mathcal{L}\{(t+1)^2\} = \mathcal{L}\{t^2 + 2t + 1\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

(d) $\mathcal{L}\{\sin(t+2\pi)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

Problem 10. Find the inverse Laplace transform of

(a). $F(s) = \frac{6}{(s-1)^4}$, **(b).** $F(s) = \frac{s+1}{s^2+2s+10}$,
(c). $F(s) = \frac{6s^2 - 13s + 2}{s(s-1)(s-6)}$, **(d.)** $F(s) = \frac{s+1}{s^2+4}$.

(a) $\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\} = e^t \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = e^t \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = e^t t^3$

(b) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$

$= e^{-t}\cos 3t$

(c) $\frac{6s^2 - 13s + 2}{s(s-1)(s-6)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-6}$

$= \frac{A(s-1)(s-6) + Bs(s-6) + Cs(s-1)}{s(s-1)(s-6)} \Rightarrow$

$$A(s-1)(s-6) + Bs(s-6) + Cs(s-1) = 6s^2 - 13s + 2$$

$s=0$: $6A = 2 \Rightarrow A = 1/3$

$s=1$: $-5B = -5 \Rightarrow B = 1$

$s=6$: $30C = 140 \Rightarrow C = 14/3 \Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s} + \frac{1}{s-1} + \frac{14}{3}\frac{1}{s-6}\right\} = \frac{1}{3} + e^{t} + \frac{14}{3}e^{6t}$$

(d) $$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}\right\}$$
$$= \cos 2t + \frac{1}{2}\sin 2t$$

Problem 11. Use Laplace transform to find the solution of an initial value problem

(a). $y'' - y' - 2y = 0,\ y(0) = -2,\ y'(0) = 5,$
(b). $y'' + 4y = 4t^2 - 4t + 10,\ y(0) = 0,\ y'(0) = 3.$

(a) $s^2 Y + 2s - 5 - (sY + 2) - 2Y = 0$

$(s^2 - s - 2)Y = 7 - 2s$

$$Y = \frac{7-2s}{s^2-s-2} = \frac{7-2s}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{A(s-2)+B(s+1)}{(s+1)(s-2)}$$

$A(s-2) + B(s+1) = 7 - 2s$

$s=2$: $3B = 3 \Rightarrow B = 1$

$s=-1$: $-3A = 9 \Rightarrow A = -3$

$$\Rightarrow Y = -\frac{3}{s+1} + \frac{1}{s-2}$$
$$\Downarrow$$
$$y = \mathcal{L}^{-1}\left\{-\frac{3}{s+1} + \frac{1}{s-2}\right\}$$

$$= -3e^{-t} + e^{2t}$$

(b) $s^2Y - 3 + 4Y = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s}$

$$(s^2+4)Y = 3 + \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} = \frac{3s^3 + 8 - 4s + 10s^2}{s^3}$$

$$Y = \frac{3s^3 + 10s^2 - 4s + 8}{s^3(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4}$$

$$= \frac{As^2(s^2+4) + Bs(s^2+4) + C(s^2+4) + s^3(Ds+E)}{s^3(s^2+4)}$$

$$\Rightarrow As^2(s^2+4) + Bs(s^2+4) + C(s^2+4) + s^3(Ds+E)$$
$$= 3s^3 + 10s^2 - 4s + 8 \Rightarrow$$

$$\cancel{As^4} + \cancel{4As^2} + \cancel{Bs^3} + \cancel{4Bs} + \cancel{Cs^2} + \cancel{4C} + \cancel{Ds^4} + \cancel{Es^3} = 3s^3 + 10s^2 - 4s + 8$$

$$(A+D)s^4 + (B+E)s^3 + (4A+C)s^2 + 4Bs + 4C = 3s^3 + 10s^2 - 4s + 8$$

$$\Rightarrow \quad A + D = 0 \Rightarrow D = -2$$
$$B + E = 3 \Rightarrow E = 4$$
$$4A + C = 10 \Rightarrow 4A = 8 \Rightarrow A = 2$$
$$4B = -4 \Rightarrow B = -1$$
$$4C = 8 \Rightarrow C = 2 \quad \Rightarrow$$

$$Y(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} + \frac{4-2s}{s^2+4} \Rightarrow$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{4-2s}{s^2+4}\right\}$$

$$= 2 - t + t^2 + 2\sin t - 2\cos t.$$

Problem 12. Find the Laplace transform of a given function

$$\textbf{(a).}\ y(t) = t^2\,\mathcal{U}(t-2)\,, \quad \textbf{(b).}\ y(t) = (\sin t + e^{3t})\,\mathcal{U}(t-2\pi).$$

(a) $\mathcal{L}\{t^2 u(t-2)\} = \mathcal{L}\{(t-2+2)^2 u(t-2)\}$

$= e^{-2s}\mathcal{L}\{(t+2)^2\} = e^{-2s}\mathcal{L}\{t^2+2t+4\}$

$= e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}\right)$

(b) $\mathcal{L}\{(\sin t + e^{3t})u(t-2\pi)\} = \mathcal{L}\{(\sin(t-2\pi$

$+2\pi) + e^{3(t-2\pi+2\pi)})u(t-2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)$

$+ e^{3(t+2\pi)}\} = e^{-2\pi s}\mathcal{L}\{\sin t + e^{6\pi}e^{3t}\}$

$= e^{-2\pi s}\left(\frac{1}{s^2+1} + \frac{e^{6\pi}}{s-3}\right)$

Problem 13. Solve the given differential equation

$$\textbf{(a).}\ x^2y'' - 7xy' + 41y = 0\,,\ x > 0, \quad \textbf{(b).}\ t^2x'' + 3tx' - 4x = 0\,,\ t > 0,$$
$$\textbf{(c).}\ x^2y'' - 5xy' + 10y = 0\,,\ x > 0, \quad \textbf{(d).}\ x^2y'' + 3xy' + y = 0\,,\ x > 0.$$

(a) $r(r-1) - 7r + 41 = 0 \Rightarrow r^2 - 8r + 41 = 0$

$r = \frac{8 \pm \sqrt{64 - 4\cdot 41}}{2} = \frac{8 \pm \sqrt{-100}}{2} = 4 \pm 5i$

$\Rightarrow y = x^4 \left(c_1 \cos(5 \ln x) + c_2 \sin(5 \ln x) \right)$

(b) $r(r-1) + 3r - 4 = 0 \Rightarrow r^2 + 2r - 4 = 0$

$r = \frac{-2 \pm (4+16)^{1/2}}{2} = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2}$

$= -1 \pm \sqrt{5} \Rightarrow$

$x(t) = c_1 t^{-1-\sqrt{5}} + c_2 t^{-1+\sqrt{5}}$

(c) $r(r-1) - 5r + 10 = 0 \Rightarrow r^2 - 6r + 10 = 0$

$r = \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$

$y(x) = x^3 \left(c_1 \cos(\ln x) + c_2 \sin(\ln x) \right)$

(d) $r(r-1) + 3r + 1 = 0 \Rightarrow r^2 + 2r + 1 = 0$

$\Rightarrow r = -1 \Rightarrow y(x) = c_1 x^{-1} + c_2 x^{-1} \ln x$

Problem 14. Solve the initial value problem

$$x^2 y'' + xy' + y = 0, \ y(1) = 1, \ y'(1) = 2.$$

$r(r-1) + r + 1 = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i$

$\Rightarrow y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

$$y'(x) = \left(-c_1 \sin(\ln x) + c_2 \cos(\ln x)\right) \frac{1}{x}$$

$$\Rightarrow \quad 1 = y(1) = c_1 \cos(\ln 1) + c_2 \sin(\ln 1) = c_1$$

$$2 = y'(1) = \frac{1}{1}\left(-c_1 \sin(\ln 1) + c_2 \cos(\ln 1)\right) = c_2$$

$$\Rightarrow \quad y(x) = \cos(\ln x) + 2\sin(\ln x)$$

Problem 15. Solve the system of linear differential equations by any method you wish

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \mathbf{x}.$$

Use Laplace Transform:

$$\begin{cases} x_1' = x_1 + 3x_2 \\ x_2' = 3x_1 + x_2 \end{cases} \Rightarrow \begin{cases} \mathcal{L}\{x_1'\} = \mathcal{L}\{x_1\} + 3\mathcal{L}\{x_2\} \\ \mathcal{L}\{x_2'\} = 3\mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} \end{cases}$$

$$\begin{cases} s\underline{\overline{X}}_1 - x_1(0) = \underline{\overline{X}}_1 + 3\underline{\overline{X}}_2 \\ s\underline{\overline{X}}_2 - x_2(0) = 3\underline{\overline{X}}_1 + \underline{\overline{X}}_2 \end{cases} \Rightarrow \begin{cases} (s-1)\underline{\overline{X}}_1 - 3\underline{\overline{X}}_2 = x_1(0) \\ -3\underline{\overline{X}}_1 + (s-1)\underline{\overline{X}}_2 = x_2(0) \end{cases}$$

$$\Rightarrow \quad \begin{aligned} 3(s-1)\underline{\overline{X}}_1 - 9\underline{\overline{X}}_2 &= 3x_1(0) \\ -3(s-1)\underline{\overline{X}}_1 + (s-1)^2\underline{\overline{X}}_2 &= (s-1)x_2(0) \end{aligned}$$

$$\Rightarrow \quad \left((s-1)^2 - 9\right)\underline{\overline{X}}_2 = 3x_1(0) + (s-1)x_2(0)$$

$$(s^2 - 2s - 8)\underline{\overline{X}}_2 = 3x_1(0) + (s-1)x_2(0)$$

$$(s-4)(s+2)\underline{X}_2 = 3x_1(0) + (s-1)x_2(0)$$

$$\underline{X}_2 = \frac{3x_1(0) + (s-1)x_2(0)}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

$$\Rightarrow A(s+2) + B(s-4) = 3x_1(0) + (s-1)x_2(0)$$

$$s=-2: \quad -6B = 3(x_1(0) - x_2(0))$$

$$\Rightarrow B = -\frac{1}{2}(x_1(0) - x_2(0))$$

$$s=4: \quad 6A = 3(x_1(0) + x_2(0))$$

$$\Rightarrow A = \frac{1}{2}(x_1(0) + x_2(0))$$

$$\Rightarrow \underline{X}_2 = \frac{1}{2}(x_1(0) + x_2(0))\frac{1}{s-4} - \frac{1}{2}(x_1(0) - x_2(0))\frac{1}{s+2}$$

$$\Rightarrow x_2 = \mathcal{L}^{-1}\{\underline{X}_2\} = \frac{1}{2}(x_1(0) + x_2(0))e^{4t} - \frac{1}{2}(x_1(0) - x_2(0))e^{-2t}$$

Because $x_2' = 3x_1 + x_2 \Rightarrow x_1 = \frac{1}{3}(x_2' - x_2)$

$$= \frac{1}{3}\Big(4\cdot\frac{1}{2}(x_1(0) + x_2(0))e^{4t} - (-2)\frac{1}{2}(x_1(0) - x_2(0))e^{-2t} - \Big(\frac{1}{2}(x_1(0) + x_2(0))e^{4t} - \frac{1}{2}(x_1(0) - x_2(0))e^{-2t}\Big)\Big)$$

$$= \frac{1}{3}\Big(\frac{3}{2}(x_1(0) + x_2(0))e^{4t} + \frac{3}{2}(x_1(0) - x_2(0))e^{-2t}\Big)$$

$$x_1 = \frac{1}{2}(x_1(0)+x_2(0))e^{4t} + \frac{1}{2}(x_1(0)-x_2(0))e^{-2t}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_1(0)+x_2(0))e^{4t} + \frac{1}{2}(x_1(0)-x_2(0))e^{-2t} \\ \frac{1}{2}(x_1(0)+x_2(0))e^{4t} - \frac{1}{2}(x_1(0)-x_2(0))e^{-2t} \end{pmatrix}$$

$$= \frac{1}{2}(x_1(0)+x_2(0))e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(x_1(0)-x_2(0))e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= c_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$