Solutions:

Problem 1. Calculate the given quantity if $\mathbf{a} = <1, 1, -2>$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = <0, 1, -5>$:

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$,

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$,

(c) The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$,

(d) The vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

1 $$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \langle 9, 15, 3 \rangle$$

a $$\bar{a} \cdot (\bar{b} \times \bar{c}) = \langle 1, 1, -2 \rangle \cdot \langle 9, 15, 3 \rangle$$
$$= 9 + 15 - 6 = 18$$

b $$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix}$$
$$= \langle 33, 21, 6 \rangle$$

c $$\text{Comp}_{\bar{a}} \bar{b} = \bar{b} \cdot \frac{\bar{a}}{|\bar{a}|} = \langle 3, -2, 1 \rangle \cdot \frac{\langle 1, 1, -2 \rangle}{|\langle 1, 1, -2 \rangle|}$$

$$= \frac{3-2-2}{(1+1+4)^{1/2}} = -\frac{1}{\sqrt{6}}$$

(d) $$\text{Proj}_{\bar{a}}\bar{b} = (\text{Comp}_{\bar{a}}\bar{b})\frac{\bar{a}}{|\bar{a}|}$$

$$= -\frac{1}{\sqrt{6}} \cdot \frac{\langle 1,1,-2\rangle}{(1+1+4)^{1/2}} = -\frac{1}{6}\langle 1,1,-2\rangle = \langle -\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}\rangle$$

Problem 2. Given the points $A(1,0,1)$, $B(2,3,0)$, $C(-1,1,4)$, and $D(0,3,2)$, find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

$$\overline{AB} = \langle 2-1, 3-0, 0-1\rangle = \langle 1,3,-1\rangle$$

$$\overline{AC} = \langle -1-1, 1-0, 4-1\rangle = \langle -2,1,3\rangle$$

$$\overline{AD} = \langle 0-1, 3-0, 2-1\rangle = \langle -1,3,1\rangle$$

$$V = \left| \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} \right| = \left| 1 \cdot \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} \right.$$

$$\left. -1\begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} \right| = |1-9-3(-2+3)-(-6+1)|$$

$$= 6$$

Problem 3. Find both parametric and symmetric equations of the line that satisfies the given conditions.

(a) Passing through $(1, 2, 4)$ and in the direction of $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$,

(b) Passing through $(-6, -1, 0)$ and $(2, -3, 5)$.

(a) $\begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = 4 + 3t \end{cases}$ $\qquad \dfrac{x-1}{2} = \dfrac{y-2}{-1} = \dfrac{z-4}{3}$

(b) $\bar{v} = \langle 2-(-6), -3-(-1), 5-0 \rangle = \langle 8, -2, 5 \rangle$

$\begin{cases} x = -6 + 8t \\ y = -1 - 2t \\ z = 5t \end{cases}$ $\qquad \dfrac{x+6}{8} = \dfrac{y+1}{-2} = \dfrac{z}{5}$

Problem 4. Find an equation of the plane that satisfies the given conditions

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(a) Passing through $(-4, 1, 2)$ and parallel to the plane $x + 2y + 5z = 3$,

(b) Passing through $(-1, 2, 0)$, $(2, 0, 1)$, and $(-5, 3, 1)$.

A B C

(a) $\bar{v} = \langle 1, 2, 5 \rangle \Rightarrow (x+4) + 2(y-1) + 5(z-2) = 0$

or $x+2y+5z=8$.

(b) $\overline{AC}, \overline{BC}$ are $\parallel$ to the plane

$\Rightarrow \quad \bar{v} = \overline{AC} \times \overline{BC}$ is $\perp$ to the plane:

$\overline{AC} = \langle -4,1,1\rangle, \quad \overline{BC} = \langle -7,3,0\rangle$

$$\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -4 & 1 & 1 \\ -7 & 3 & 0 \end{vmatrix} = \langle -3,7,-5\rangle \Rightarrow$$

$$-3(x+1)+7(y-2)-5(z-0)=0, \text{ or}$$

$$-3x+7y-5z=17$$

<u>Problem 5</u>. Determine whether the lines given by the symmetric equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and

$$\frac{x+1}{6} = \frac{y-3}{-1} = \frac{z+5}{2}$$

are parallel, skew, or intersecting. If the lines are intersecting, find the point of intersection and the angle between the lines.

The first line is $\parallel$ to $\langle 2,3,4\rangle$;
the second line is $\parallel$ to $\langle 6,-1,2\rangle$

These vectors are not $\parallel$ $\Rightarrow$ lines are either intersecting or skew. Rewrite the equations in parametric form:

$$L1: \begin{cases} x=1+2t \\ y=2+3t \\ z=3+4t \end{cases} \qquad L2: \begin{cases} x=-1+6s \\ y=3-s \\ z=-5+2s \end{cases}$$

At the intersection point:

$$\begin{cases} 1+2t=-1+6s \\ 2+3t=3-s \quad \rightarrow \quad s=1-3t \\ 3+4t=-5+2s \end{cases}$$

; substitute into the 1st and 3rd equations:

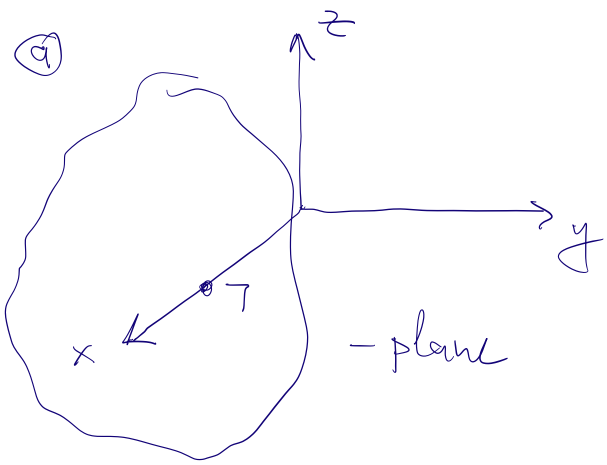
$$\left.\begin{aligned} 1+2t&=-1+6(1-3t) \Rightarrow t=1/5 \\ 3+4t&=-5+2(1-3t) \Rightarrow t=-3/5 \end{aligned}\right\}$$

the system has no solution $\Rightarrow$ lines are skew.

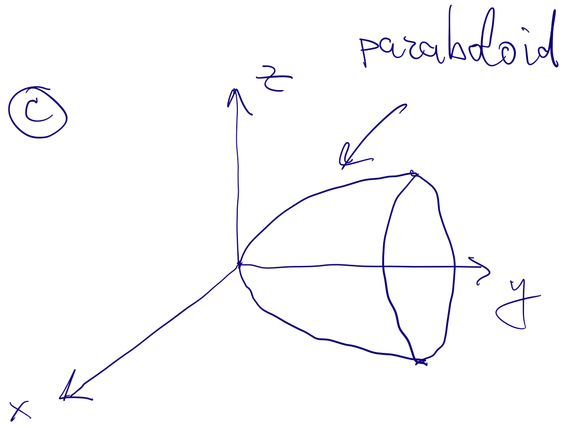
Problem 6. Identify and sketch the graph of each surface

(a) $x=7$, (b) $y=z$, (c) $y=x^2+z^2$.
(d) $x^2-y^2+z^2=0$, (e) $x^2+z^2=1$.

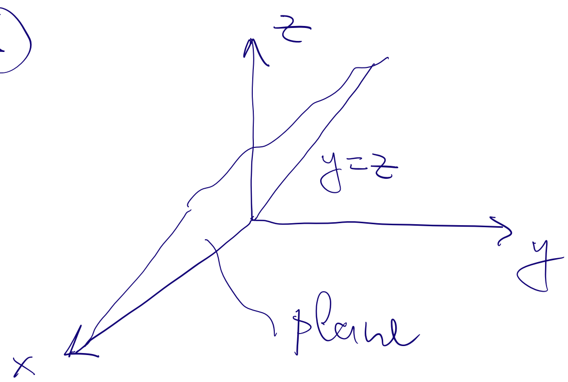
(a)



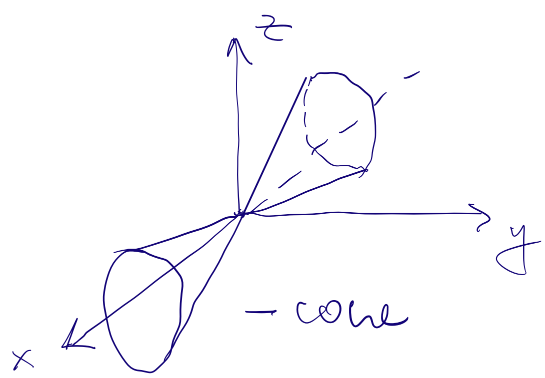
(c)



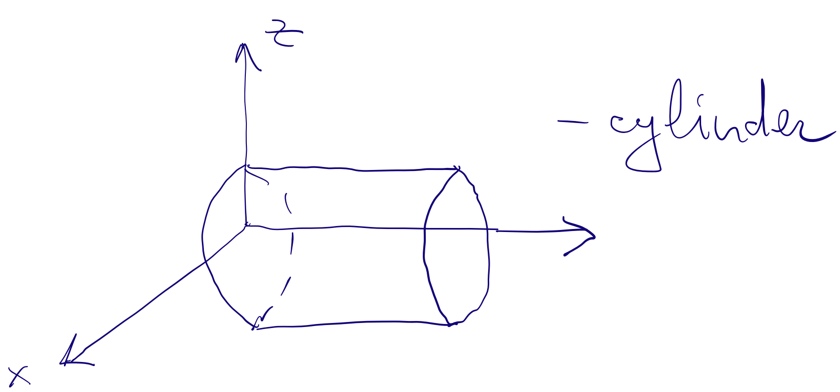
(b)



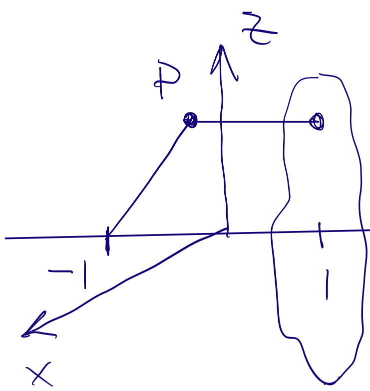
(d) $\Rightarrow x=\pm\sqrt{y^2+z^2}$



(e)



Problem 7. A surface consists of all points $\boldsymbol{P}$ such that the distance from $\boldsymbol{P}$ to the plane $y = 1$ is twice the distance from $\boldsymbol{P}$ to the point $(0, -1, 0)$. Find an equation for this surface and identify it.



* distance from $P(x,y,z)$ to $(0,-1,0)$ is

$$\left((x-0)^2+(y+1)^2+(z-0)^2\right)^{1/2}$$
$$=\left(x^2+(y+1)^2+z^2\right)^{1/2}$$

* distance from P to the plane $|y-1|$

$\Rightarrow$ Surface has the equation:

$$|y-1| = 2\left(x^2+(y+1)^2+z^2\right)^{1/2}$$

$$\Rightarrow (y-1)^2 = 4\left(x^2+(y+1)^2+z^2\right)$$

$$y^2-2y+1 = 4\left(x^2+y^2+2y+1+z^2\right)$$

$$\Rightarrow 4x^2+4y^2+8y+4+4z^2-y^2+2y-1=0$$

$$4x^2+3y^2+10y-1+4z^2=0$$

$$4x^2+3\left(y^2+\frac{10}{3}y\right)+4z^2=1$$

$$4x^2+3\left(y^2+\frac{10}{3}y+\frac{25}{9}-\frac{25}{9}\right)+4z^2=1$$

$$4x^2+3\left(y+\frac{5}{3}\right)^2+4z^2=1+\frac{25}{3}=\frac{28}{3}$$

→ an ellipsoid.

<u>Problem 8</u>. Sketch the curve with vector function

$$\mathbf{r}(t)=2\mathbf{i}+\sin t\,\mathbf{j}+\cos t\,\mathbf{k}\,.$$

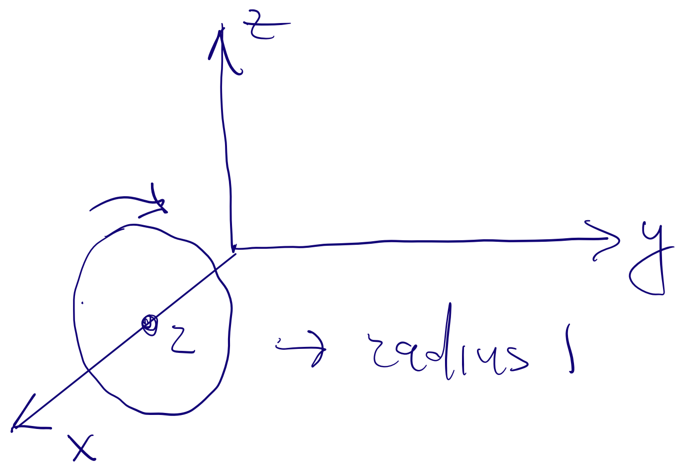
$x(t)=2$

$y(t)=\sin t$

$z(t)=\cos t$

$\Rightarrow$ eliminate t:

$$y^2(t)+z^2(t)=1$$



Problem 9. The helix $\mathbf{r_1}(t) = < \cos t, \sin t, t >$ intersects the curve $\mathbf{r_2}(t) = < 1 + t, t^2, t^3 >$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves as well as a curvature of each curve at the point of their intersection. Also, find the unit normal vector to $\boldsymbol{r}_1(t)$ when t=0.

the curves intersect when $t=0$; the direction of $\bar{r}_1(t)$ at $t=0$ is $\bar{T}_1(0)$ and the direction of $\bar{r}_2(t)$ at $t=0$ is $\bar{T}_2(0)$

$$\bar{T}_1(0) = \frac{\bar{r}_1'(0)}{|\bar{r}_1'(0)|} = \frac{\langle 0,1,1\rangle}{(0+1+1)^{1/2}} = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

because $\bar{r}_1'(t) = \langle -\sin t, \cos t, 1\rangle \Rightarrow$
$\bar{r}_1'(0) = \langle 0,1,1\rangle$

$$\bar{T}_2(0) = \frac{\bar{r}_2'(0)}{|\bar{r}_2'(0)|} = \frac{\langle 1,0,0\rangle}{(1+0+0)^{1/2}} = \langle 1,0,0\rangle$$

because $\bar{r}_2'(t) = \langle 1, 2t, 3t^2\rangle \Rightarrow$
$\bar{r}_2'(0) = \langle 1,0,0\rangle$

$\Rightarrow$ cosine of the angle between $\bar{T}_1(0)$ and $\bar{T}_2(0)$ is

$$\cos\theta = \frac{\bar{T}_1(0)\cdot\bar{T}_2(0)}{|\bar{T}_1(0)|\,|\bar{T}_2(0)|} = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \cdot \langle 1,0,0\rangle = 0$$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

To find the unit normal vector to $\bar{r}_1(t)$, first compute $\bar{T}_1(t) = \dfrac{\bar{r}_1'(t)}{\|\bar{r}_1'(t)\|} = \dfrac{\langle -\sin t, \cos t, 1\rangle}{(\sin^2 t + \cos^2 t + 1)^{1/2}}$

$$= \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

$$\Rightarrow \bar{T}_1'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$$

$$\bar{T}_1'(0) = \frac{1}{\sqrt{2}} \langle -\cos 0, -\sin 0, 0 \rangle = \langle -\tfrac{1}{\sqrt{2}}, 0, 0 \rangle$$

$$\bar{N}_1(0) = \frac{\bar{T}_1'(0)}{\|\bar{T}_1'(0)\|} = \frac{\langle -\frac{1}{\sqrt{2}}, 0, 0 \rangle}{1/\sqrt{2}} = \langle -1, 0, 0 \rangle$$

To find curvatures, we need

$$\bar{r}_1'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad \bar{r}_2'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\bar{r}_1''(t) = \langle -\cos t, -\sin t, 0 \rangle, \quad \bar{r}_2''(t) = \langle 0, 2, 6t \rangle$$

$$\Rightarrow \bar{r}_1'(0) = \langle 0, 1, 1 \rangle, \quad \bar{r}_2'(0) = \langle 1, 0, 0 \rangle$$

$$\bar{r}_1''(0) = \langle -1, 0, 0 \rangle, \quad \bar{r}_2''(0) = \langle 0, 2, 0 \rangle$$

$$\Downarrow$$

$$k_1(0)=\frac{|\bar{r}_1'(0)\times\bar{r}_1''(0)|}{|\bar{r}_1'(0)|^3} \qquad k_2(0)=\frac{|\bar{r}_2'(0)\times\bar{r}_2''(0)|}{|\bar{r}_2'(0)|^3}$$

$$=\frac{|\langle 0,-1,1\rangle|}{(\sqrt{2})^3}=\frac{\sqrt{2}}{(\sqrt{2})^3}=\frac{1}{2}; \qquad =\frac{|\langle 0,0,2\rangle|}{1^3}=2$$

because

$$\bar{r}_1'(0)\times\bar{r}_1''(0)=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 0 & 1 & 1\\ -1 & 0 & 0\end{vmatrix}=\langle 0,-1,1\rangle$$

$$\bar{r}_2'(0)\times\bar{r}_2''(0)=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 0 & 0\\ 0 & 2 & 0\end{vmatrix}=\langle 0,0,2\rangle$$

Problem 10. A particle starts at the origin with initial velocity $<1,2,1>$ and its acceleration is $\mathbf{a}(t)=<t,1,t^2>$. Find its position function.

$$\bar{v}(t)=\int\langle t,1,t^2\rangle\,dt=\langle t^2/2,\,t,\,t^3/3\rangle+\bar{c}$$

$$\langle 1,2,1\rangle=\bar{v}(0)=\langle 0,0,0\rangle+\bar{c}$$

$$\Rightarrow c=\langle 1,2,1\rangle \text{ and}$$

$$\overline{v}(t) = \left\langle t^2/2 + 1,\ t+2,\ t^3/3 + 1 \right\rangle$$

$$F(t) = \int \overline{v}(t)\,dt = \int \left\langle t^2/2 + 1,\ t+2,\ t^3/3 + 1 \right\rangle dt$$

$$= \left\langle \frac{t^3}{6} + t,\ \frac{t^2}{2} + 2t,\ \frac{t^4}{12} + t \right\rangle + \overline{D}$$

$$\overline{0} = F(0) = \langle 0, 0, 0 \rangle + \overline{D} \Rightarrow \overline{D} = \overline{0}$$

$$\Rightarrow \quad F(t) = \left\langle \frac{t^3}{6} + t,\ \frac{t^2}{2} + 2t,\ \frac{t^4}{12} + t \right\rangle$$