

These problems provide a sample of typical problems you are expected to be able to solve.

1. Vector Operations. Let  $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$  and  $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ . Find

- (a)  $\mathbf{u} \cdot \mathbf{v}$
- (b)  $\mathbf{u} \times \mathbf{v}$
- (c)  $3\mathbf{u} \cdot \mathbf{v} - (2\mathbf{u} + 3\mathbf{v}) \times \mathbf{v}$
- (d) the angle between  $\mathbf{u}$  and  $\mathbf{v}$
- (e)  $\text{proj}_{\mathbf{v}} \mathbf{u}$
- (f)  $\text{proj}_{\mathbf{u}} \mathbf{v}$

2. Lines, Planes and Quadric Surfaces

- (a) Find the equation of the line going through the points (1,2,0) and (4,5,6).
- (b) Find the equation of the plane going through the points (2, 0, -1), (0, 4, 1) and (-1, -1, 0).
- (c) Find the point of intersection of the line  $\mathbf{r}(t) = \langle 2t + 1, 3t - 2, t + 1 \rangle$  and the plane  $x - 3y + 2z = 7$ .
- (d) Find the angle between the line  $\mathbf{r}(t) = \langle 2t + 1, 3t - 2, t + 1 \rangle$  and the plane  $x - 3y + 2z = 7$ .
- (e) Find the distance between the point  $P = (3, -2, 6)$  and the plane  $-2x + y - 3z = 4$ .
- (f) Use intercepts to sketch the plane  $2x + 4y + 8z = 16$ .
- (g) Sketch the surface  $z = 5 - (x^2 + y^2)$ . What type of quadric surface is this?
- (h) Sketch the surface  $z = -3 + \sqrt{x^2 + y^2}$ . What type of quadric surface is this?
- (i) Sketch the surface  $x^2 + 4y^2 + 9z^2 = 36$ . What type of quadric surface is this?
- (j) What is the difference between  $x^2 + y^2 + z^2 = 1$  and  $x^2 + y^2 + z^2 \leq 1$ ?

3. Motion along a 1D curve in  $R^3$ . Let an object follow the path  $\mathbf{r}(t) = \langle 2 \cos 4t, 6t - 2, 2 \sin 4t \rangle$ . Find

- (a) the velocity, speed and acceleration
- (b) the unit tangent and normal vectors
- (c) the curvature
- (d) the tangential and normal components of acceleration

4. Partial derivatives

(a) For the functions below, evaluate all first and second derivatives, and simplify as much as possible.

- i.  $f(x, y) = \ln \left( \frac{x}{y^2} + 1 \right)$
- ii.  $f(x, y, z) = e^{xy^2/z}$
- iii.  $f(x, y, z) = \frac{(z + y \sin x)^{4/9}}{(x^2 + y^3 + z^4)^{1/3}}$

- (b) Let  $f(x, y) = x^2y$ . Use the chain rule to find the polar derivatives  $\frac{\partial f}{\partial r}$  and  $\frac{\partial^2 f}{\partial r \partial \theta}$ .
- (c) Show that  $u(x, t) = e^{-D\pi^2 t} \sin(\pi x)$  satisfies the diffusion equation  $u_t = Du_{xx}$ , where  $D$  is a constant.
- (d) Find the tangent plane to  $f(x, y) = \ln(x^2/y)$  at the point  $(3, 4)$ .
- (e) Use a linear approximation to estimate  $Q = \frac{2.02^4}{1.97^3 + 4}$ .
- (f) Find the rate of change of the function  $f(x, y, z) = x^2y + z^3$  at the point  $(1, 2, 3)$  in the direction given by  $\mathbf{v} = \langle -1, 4, 5 \rangle$ .
- (g) Find the upward pointing normal vector to the curve  $y = e^x$  in  $R^2$ .
- (h) Find the downward pointing normal vector to the surface  $z = x^2 + y^2$ .
- (i) Find and classify the critical points of  $f(x, y) = x^3 - 3x^2y + y^2$ .
- (j) Find the extreme values of  $f(x, y) = x^2 + 2xy^2$  on the domain  $x^2 + y^2 \leq 2$ .
- (k) Find the maximum and minimum values of  $f(x, y) = ye^x$  along the constraint curve  $x^2 + 2y^2 = 4$ .
- (l) Find the maximum value of  $f(x, y) = x^2 - 6y^2 + 2$  along the constraint curve  $2x + 3y = 5$ .

## 5. Integrals

- (a) Evaluate  $\iint_D y(x-3)^3 + 1 \, dA$ , where  $D = \{(x, y) | 1 \leq x \leq 4, x-3 \leq y \leq 1\}$
- (b) Consider  $\int_0^8 \int_{\sqrt[3]{x}}^2 e^{x^4} \, dx dy$ . Change the order of integration then evaluate the integral.
- (c) Evaluate  $\iint_D 1 + x^2 + y^2 \, dA$ , where  $D$  is the circle of radius 2 centered at the origin.
- (d) Find the surface area of the portion of the surface  $z = 10 - 2x^2 - 2y^2$  that lies above the  $xy$ -plane.
- (e) Evaluate  $\iiint_D xy \, dV$ , where  $D$  is the region in the first octant bounded by the coordinate planes and  $x + 2y + 3z = 6$ .
- (f) Evaluate  $\iint_D (x^2 + y^2)^2 \, dA$ , where  $D$  is the region bounded above by  $z = 10 - x^2 - y^2$  and below by  $z = 8$ .
- (g) Modify the standard cylindrical coordinates to evaluate  $\iiint_D (x^2 + z^2) \, dA$ , where  $D$  is the cylinder of radius 1 with axis centered along the  $y$ -axis, with  $1 \leq y \leq 3$ .
- (h) Use spherical coordinates to evaluate  $\iiint_D x^2 + y^2 + z^2 \, dA$ , where  $D$  is the region bounded above by  $x^2 + y^2 + z^2 = 25$  and below by  $z = \sqrt{x^2 + y^2}$ .

- (i) Use a change of variables to evaluate  $\iint_D xy \, dA$ , where  $D$  is the diamond-shaped domain bounded by  $y = x + 1$ ,  $y = 1 - x$ ,  $y = x - 1$  and  $y = -1 - x$ .

## 6. Vector Calculus

- (a) Evaluate  $I = \int_C x + 2y + 3z \, ds$ , where  $C = C_1 + C_2$ . The first segment of the path,  $C_1$ , is the arc of the circle  $x^2 + y^2 = 1$  lying in the  $xy$ -plane, beginning at the point  $(1, 0, 0)$  and ending at the point  $(0, 1, 0)$ . The second segment of the path,  $C_2$ , is the arc of the circle  $y^2 + z^2 = 1$  lying in the  $yz$ -plane, beginning at the point  $(0, 1, 0)$  and ending at the point  $(0, 0, 1)$ .
- (b) Find the work done (the vector line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ ) by a particle moving through the vector field  $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} - x\mathbf{k}$  along the path  $C = C_1 + C_2$ , where  $C_1$  is the straight line segment from  $(1, 1, 1)$  to  $(2, 3, 4)$  and  $C_2$  is the straight line segment from  $(2, 3, 4)$  to  $(0, -1, 5)$ .
- (c) Show that the vector field  $\mathbf{F} = \langle e^y + \sec x \tan x \tan y + y \cos(xy) + 2x, xe^y + \sec x \sec^2 y + x \cos(xy) - 3y^2 \rangle$  is conservative, and find a potential for it.
- (d) The vector field  $\mathbf{F} = \left\langle y^2 z^4 + \frac{1}{x} + \frac{z}{y} e^{xz}, 2xyz^4 + \frac{1}{y} - \frac{1}{y^2} e^{xz}, 4xy^2 z^3 - \frac{1}{z} + \frac{x}{y} e^{xz} \right\rangle$  is conservative (you don't need to verify it). Find a potential for  $\mathbf{F}$ .
- (e) Use Green's Theorem to evaluate the vector line integral of  $\mathbf{F} = \left\langle \frac{y^2}{2} + \tan(e^{x^2}), \ln(y^4) - \frac{x^2}{2} \right\rangle$  over the closed curve  $C$ , which is the circle of radius 2 centered at the origin oriented in the counterclockwise direction.
- (f) Find the divergence and curl of  $\mathbf{F} = \sin(x^2 + y)\mathbf{i} - e^{xz}\mathbf{j} + \ln(x + 2y + 3z)\mathbf{k}$ .
- (g) Find the flux,  $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$ , of the vector field  $\mathbf{F} = \mathbf{i} - z\mathbf{j} + x\mathbf{k}$  across the surface  $S$ , where  $S$  is the portion of the plane  $x + 2y + z = 2$  that lies in the first octant, with upward pointing normal.
- (h) Use Stokes' Theorem to compute  $\iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F} = \langle xy, xz, yz \rangle$  and  $S$  is the part of the sphere  $x^2 + y^2 + z^2 = 9$  that lies inside the cylinder  $x^2 + y^2 = 1$  and is above the  $xy$ -plane.
- (i) Use the Divergence Theorem to find the net flux of  $\mathbf{F} = \langle x, 2y, 4z \rangle$  out of the sphere of radius 3 centered at the origin.