1. Vector Operations. Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find

$\langle 1,-2,3\rangle$ $\langle -2,1,1\rangle$

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $\mathbf{u} \times \mathbf{v}$

(c) $3\mathbf{u} \cdot \mathbf{v} - (2\mathbf{u} + 3\mathbf{v}) \times \mathbf{v}$

(d) the angle between $\mathbf{u}$ and $\mathbf{v}$

(e) $\text{proj}_{\mathbf{v}}\mathbf{u}$

(f) $\text{proj}_{\mathbf{u}}\mathbf{v}$

(a) $\bar{u}\cdot\bar{v} = 1(-2) + (-2)\cdot 1 + 3\cdot 1 = -1$

(b) $\bar{u}\times\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 3 \\ -2 & 1 & 1 \end{vmatrix} = \langle -2-3, -(1+6), 1-4\rangle = \langle -5,-7,-3\rangle$

(c) Makes no sense - cannot add a scalar to a vector.

(d) $\cos\theta = \dfrac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|} = \dfrac{-1}{(1^2+2^2+3^2)^{1/2}(2^2+1^2+1^2)^{1/2}} = -\dfrac{1}{\sqrt{14}\sqrt{6}} = -\dfrac{1}{\sqrt{84}}$

$\theta = \cos^{-1}\left(-\dfrac{1}{\sqrt{84}}\right)$

(e-f) $\text{proj}_{\bar{v}}\bar{u} = \dfrac{\bar{u}\cdot\bar{v}}{|\bar{v}|^2}\bar{v} = \dfrac{-1}{6}\langle -2,1,1\rangle$

$\text{proj}_{\bar{u}}\bar{v} = \dfrac{\bar{u}\cdot\bar{v}}{|\bar{u}|^2}\bar{u} = -\dfrac{1}{14}\langle 1,-2,3\rangle$

2. Lines, Planes and Quadric Surfaces

(a) Find the equation of the line going through the points (1,2,0) and (4,5,6).

(b) Find the equation of the plane going through the points $(2, 0, -1)$, $(0, 4, 1)$ and $(-1, -1, 0)$.

(c) Find the point of intersection of the line $\mathbf{r}(t) =< 2t + 1, 3t - 2, t + 1 >$ and the plane $x - 3y + 2z = 7$.

(d) Find the angle between the line $\mathbf{r}(t) =< 2t+1, 3t-2, t+1 >$ and the plane $x-3y+2z = 7$.

(e) Find the distance between the point $P = (3, -2, 6)$ and the plane $-2x + y - 3z = 4$.

(f) Use intercepts to sketch the plane $2x + 4y + 8z = 16$.

(g) Sketch the surface $z = 5 - (x^2 + y^2)$. What type of quadric surface is this?

(h) Sketch the surface $z = -3 + \sqrt{x^2 + y^2}$. What type of quadric surface is this?

(i) Sketch the surface $x^2 + 4y^2 + 9z^2 = 36$. What type of quadric surface is this?

(j) What is the difference between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 \leq 1$?

(a) $\bar{v} = \langle 4,5,6\rangle - \langle 1,2,0\rangle = \langle 3,3,6\rangle$

$$\Rightarrow \quad \frac{x-1}{3} = \frac{y-2}{3} = \frac{z}{6}$$

(b) $A(2,0,-1)$, $B(0,4,1)$, $C(-1,-1,0)$

$\overline{AC} = \langle -1-2, -1-0, 0-(-1)\rangle = \langle -3,-1,1\rangle$

$\overline{AB} = \langle 0-2, 4-0, 1-(-1)\rangle = \langle -2,4,2\rangle$

$$\bar{V} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -3 & -1 & 1 \\ -2 & 4 & 2 \end{vmatrix} = \langle -6, 4, -10\rangle$$

use A and $\bar{V}$:

$$-6(x-2) + 4(y-0) - 10(z+1) = 0$$

$$\Rightarrow \quad -6x + 4y - 10z = 22$$

(c) $\langle 2t+1, 3t-2, t+1\rangle$ intersects $x - 3y + 2z = 7$

when t satisfies

$$(2t+1) - 3(3t-2) + 2(t+1) = 7$$

$$2t+1-9t+6+2t+2 = 7$$

$$-5t = -2 \Rightarrow t = \frac{2}{5}$$

$\Rightarrow$ the intersection point is

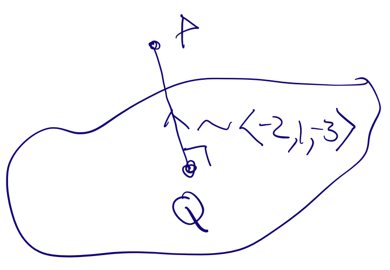
$$\langle 2\cdot\tfrac{2}{5}+1, 3\tfrac{2}{5}-2, \tfrac{2}{5}+1\rangle = \langle \tfrac{9}{5}, -\tfrac{4}{5}, \tfrac{7}{5}\rangle$$

(d) $\langle 2,3,1\rangle \parallel$ line, $\langle 1,-3,2\rangle \perp$ plane $\Rightarrow$

angle between the line and the vector $\perp$ to the plane is

$$\cos^{-1}\left(\frac{\langle 2,3,1\rangle\cdot\langle 1,-3,2\rangle}{|\langle 2,3,1\rangle||\langle 1,-3,2\rangle|}\right) = \cos^{-1}\left(\frac{2-9+2}{14}\right) = \cos^{-1}\left(-\frac{5}{14}\right)$$

(e) $P(3,-2,6)$, $-2x+y-3z=4$



Line through P $\perp$ to the plane is
$$\begin{cases} x = 3-2t \\ y = -2+t \\ z = 6-3t \end{cases}$$

intersects the plane when t satisfies

$$-2(3-2t)+(-2+t)-3(6-3t)=4$$
$$-6+4t-2+t-18+9t=4$$
$$14t = 30 \Rightarrow t = \frac{30}{14} = \frac{15}{7}$$

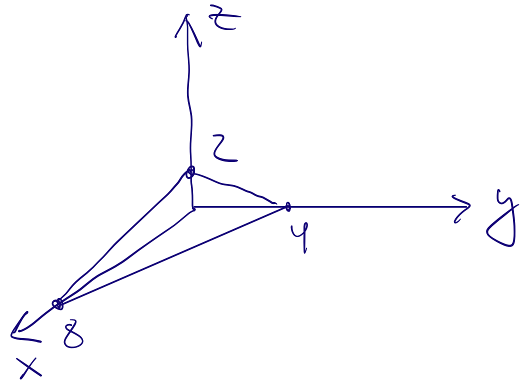
Intersection point Q: $\left(3-2\frac{15}{7}, -2+\frac{15}{7}, 6-3\cdot\frac{15}{7}\right)$
$= \left(-\frac{9}{7}, \frac{1}{7}, -\frac{3}{7}\right)$

distance $(P,Q) = \left(\left(3+\frac{9}{7}\right)^2 + \left(-2-\frac{1}{7}\right)^2 + \left(6+\frac{3}{7}\right)^2\right)^{1/2} = \frac{15\sqrt{14}}{7}$

(f) $2x+4y+8z=16 \Rightarrow$ $y=z=0$: $2x=16 \Rightarrow x=8$

$x=z=0$: $4y=16 \Rightarrow y=4$

$x=y=0$: $8z=16 \Rightarrow z=2$

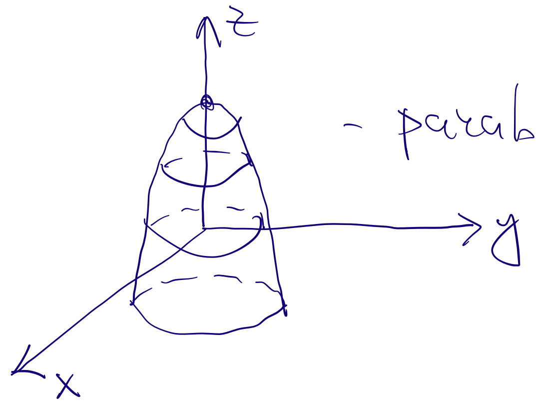


(g) $z=5-(x^2+y^2)$: Intersections by planes $z=c \Rightarrow$

$c=5-(x^2+y^2)$

$x^2+y^2=5-c$ - circles when $c \leq 5$

Intersection by the plane $x=0$: $z=5-y^2$ - parabola

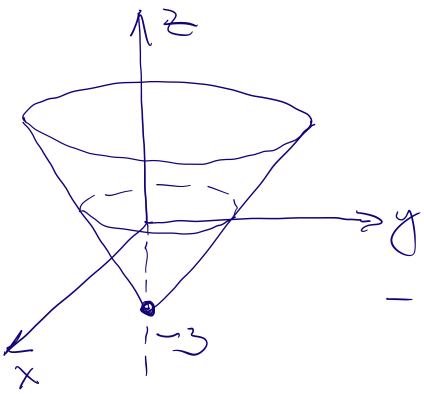


(h) $z = -3 + \sqrt{x^2+y^2} \Rightarrow$ $z = c$: $c = -3 + \sqrt{x^2+y^2}$

$x^2+y^2 = (c+3)^2$

– a circle of radius $c+3$

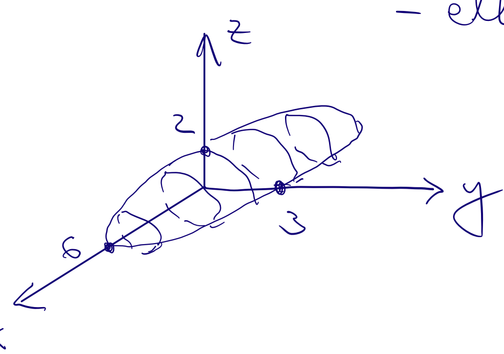
$x = 0$: $z = -3 + \sqrt{y^2} = -3 + |y|$



– a cone

(i) $x^2 + 4y^2 + 9z^2 = 36$

– ellipsoid



(j) $x^2+y^2+z^2 = 1$ – sphere of radius 1

$x^2+y^2+z^2 \leq 1$ – ball of radius 1

3. Motion along a 1D curve in R^3. Let an object follow the path $\mathbf{r}(t) = < 2\cos 4t, 6t-2, 2\sin 4t >$. Find

   (a) the velocity, speed and acceleration

   (b) the unit tangent and normal vectors

   (c) the curvature

   (d) the tangential and normal components of acceleration

(a) $\bar{v}(t) = \bar{r}'(t) = \langle -8\sin 4t, 6, 8\cos 4t \rangle$; $\bar{a}(t) = \bar{v}'(t) = \langle -32\cos 4t, 0, -32\sin 4t \rangle$

$v(t) = |\bar{r}'(t)| = ((8\sin 4t)^2 + 6^2 + (8\cos 4t)^2)^{1/2} = 10$

(b) $\bar{T}(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|} = \frac{1}{10}\langle -8\sin 4t, 6, 8\cos 4t\rangle = \langle -\frac{4}{5}\sin 4t, \frac{3}{5}, \frac{4}{5}\cos 4t\rangle$

$\bar{T}'(t) = \langle -\frac{16}{5}\cos 4t, 0, -\frac{16}{5}\sin 4t\rangle$

$\bar{N}(t) = \frac{\bar{T}'(t)}{|\bar{T}'(t)|} = \frac{\langle -\frac{16}{5}\cos 4t, 0, -\frac{16}{5}\sin 4t\rangle}{16/5} = \langle -\cos 4t, 0, -\sin 4t\rangle$

(c) $k(t) = \frac{|\bar{T}'(t)|}{|\bar{r}'(t)|} = \frac{16}{5}\frac{1}{10} = \frac{8}{25}$

(d) $a_T(t) = v'(t) = 0$

$a_N(t) = k(t)v^2(t) = \frac{8}{25}\cdot 10^2 = 32$