4a

i. $f(x,y) = \ln\left(\frac{x}{y^2}+1\right)$

$$f_x = \frac{1}{x/y^2+1}\,\frac{1}{y^2} = \frac{1}{x+y^2}\,,\quad f_y = \frac{1}{x/y^2+1}\left(-\frac{2x}{y^3}\right) = -\frac{2x}{yx+y^3}$$

$$f_{xx} = -\frac{1}{(x+y^2)^2}\,;\quad f_{xy} = -\frac{2y}{(x+y^2)^2}$$

$$f_{yy} = \frac{2x}{(yx+y^3)^2}(x+3y^2)$$

ii. $f(x,y,z) = e^{xy^2/z}$

$$f_x = e^{xy^2/z}\,\frac{y^2}{z}\,;\quad f_y = e^{xy^2/z}\,\frac{2xy}{z}\,;\quad f_z = e^{xy^2/z}\left(-\frac{xy^2}{z^2}\right)$$

$$f_{xx} = e^{xy^2/z}\,\frac{y^4}{z^2}\,;\quad f_{yy} = e^{xy^2/z}\left(\frac{2xy}{z}\right)^2 + e^{xy^2/z}\,\frac{2x}{z}\,;$$

$$f_{zz} = e^{xy^2/z}\left(\frac{xy^2}{z^2}\right)^2 + e^{xy^2/z}\,\frac{2xy^2}{z^3}\,;\quad f_{xy} = e^{xy^2/z}\,\frac{2xy}{z}\,\frac{y^2}{z} + e^{xy^2/z}\,\frac{2y}{z}$$

$$f_{xz} = e^{xy^2/z}\left(-\frac{xy^2}{z^2}\right)\frac{y^2}{z} + e^{xy^2/z}\left(-\frac{y^2}{z^2}\right);$$

$$f_{yz} = e^{xy^2/z}\left(-\frac{xy^2}{z^2}\right)\frac{2xy}{z} + e^{xy^2/z}\left(-\frac{2xy}{z^2}\right)$$

(c) Show that $u(x,t) = e^{-D\pi^2 t}\sin(\pi x)$ satisfies the diffusion equation $u_t = Du_{xx}$, where D is a constant.

$$u_t = -D\pi^2 e^{-D\pi^2 t}\sin(\pi x)\,;\quad u_x = e^{-D\pi^2 t}\,\pi\cos\pi x \quad \Rightarrow$$

$$u_{xx} = e^{-D\pi^2 t}(-\pi^2 \sin \pi x) \Rightarrow u_t = D u_{xx}$$

(d) Find the tangent plane to $f(x,y) = \ln(x^2/y)$ at the point $(3,4)$.

$$x_0 = 3,\ y_0 = 4,\ z_0 = \ln(3^2/4) = \ln(9/4)$$

$$\frac{\partial f}{\partial x} = \frac{\cancel{y}}{x^2}\cdot\frac{2x}{\cancel{y}} = \frac{2}{x} \Rightarrow \frac{\partial f}{\partial x}(x_0,y_0) = \frac{2}{3}$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2}\left(-\frac{x^2}{y^2}\right) = -\frac{1}{y} \Rightarrow \frac{\partial f}{\partial y}(x_0,y_0) = -\frac{1}{4}$$

Tangent plane: $z - \ln(9/4) = \frac{2}{3}(x-3) - \frac{1}{4}(y-4)$

$$z - \ln(9/4) = \frac{2}{3}x - 2 - \frac{1}{4}y + 1$$

$$z = \frac{2}{3}x - \frac{1}{4}y - 1 + \ln(9/4)$$

(e) Use a linear approximation to estimate $Q = \dfrac{2.02^4}{1.97^3 + 4}$.

$$Q(x,y) = \frac{x^4}{y^3+4} \Rightarrow Q(2,2) = \frac{2^4}{2^3+4} = \frac{16}{12} = \frac{4}{3}$$

Linearization at $(2,2)$:

$$Q(x,y) \sim Q(2,2) + \frac{\partial Q}{\partial x}(2,2)(x-2) + \frac{\partial Q}{\partial y}(2,2)(y-2)$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{4x^3}{y^3+4};\quad \frac{\partial Q}{\partial y} = -\frac{x^4}{(y^3+4)^2}\cdot 3y^2$$

$$\Rightarrow \frac{\partial Q}{\partial x}(2,2) = \frac{4\cdot 2^3}{2^3+4} = \frac{32}{12} = \frac{8}{3};\quad \frac{\partial Q}{\partial y}(2,2) = -\frac{2^4\cdot 3\cdot 2^2}{(2^3+4)^2}$$

$$= -\frac{\cancel{16}\cdot\cancel{3}\cdot 4}{\cancel{16}\cdot 3^{\cancel{2}}} = -\frac{4}{3}$$

$$\Rightarrow Q(x,y) \approx \frac{4}{3} + \frac{8}{3}(x-2) - \frac{4}{3}(y-2)$$

$$\Rightarrow Q(2.02, 1.97) = \frac{4}{3} + \frac{8}{3}(2.02-2) - \frac{4}{3}(1.97-2)$$

$$= \frac{4}{3}(1 + 2\cdot 0.02 + 0.03) = \frac{4}{3}\cdot 1.07 = \frac{4.28}{3}$$

(f) Find the rate of change of the function $f(x,y,z) = x^2y + z^3$ at the point $(1,2,3)$ in the direction given by $\mathbf{v} = < -1, 4, 5 >$.

$$\nabla f = \langle 2xy, x^2, 3z^2 \rangle \Rightarrow \nabla f(1,2,3) = \langle 4, 4, 27 \rangle$$

$$n = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -1, 4, 5 \rangle}{\sqrt{1+16+25}} = \frac{1}{\sqrt{42}}\langle -1, 4, 5 \rangle$$

$$D_n f = \langle 4, 4, 27 \rangle \cdot \frac{1}{\sqrt{42}}\langle -1, 4, 5 \rangle = \frac{-4+16+135}{\sqrt{42}}$$

$$= \frac{147}{\sqrt{42}}$$

(g) Find the upward pointing normal vector to the curve $y = e^x$ in R^2.

Curve $y = e^x$ is a level curve for $f(x,y) = e^x - y = 0$

Normal to this curve is $\nabla f = \langle e^x, -1 \rangle$

– this has y-component $= -1$ $\Rightarrow$ upward pointing normal is $-\nabla f = \langle -e^x, 1 \rangle$

(h) Find the downward pointing normal vector to the surface $z = x^2 + y^2$.

– this is a level surface for $f(x,y,z) = x^2+y^2-z$

Normal to the surface: $\nabla f = \langle 2x, 2y, -1 \rangle$

$\uparrow$ negative $\Rightarrow$ normal points downward.

(i) Find and classify the critical points of $f(x, y) = x^3 - 3x^2y + y^2$.

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy \ ; \ \frac{\partial f}{\partial y} = -3x^2 + 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 6y \ ; \ \frac{\partial^2 f}{\partial x \partial y} = -6x \ ; \ \frac{\partial^2 f}{\partial y^2} = 2$$

Critical points: $\begin{cases} 3x^2 - 6xy = 0 \Rightarrow x^2 - 2xy = 0 \Rightarrow x(x-2y) = 0 \\ -3x^2 + 2y = 0 \end{cases}$

$\Downarrow$

$x = 0 \ ; \ x = 2y$

$\Downarrow$ $x=0$: $0 + 2y = 0$, $y = 0$

$\Downarrow$ $x=2y$: $-3(2y)^2 + 2y = 0$

$2y(1-6y) = 0$

$\Downarrow$ $y = 0$ $\quad$ $\Downarrow$ $y = \frac{1}{6}$

$\Downarrow$ $x = 2y$ $\Downarrow$

$x = 0$ $\quad$ $y = \frac{1}{3}$

$\Leftarrow$ two critical points $(0,0)$ and $\left(\frac{1}{3}, \frac{1}{6}\right)$

$f_{xx}(0,0) = 0$;

$D(0,0) = 0$;

saddle

$f_{xx}\left(\frac{1}{3}, \frac{1}{6}\right) = 1$

$D\left(\frac{1}{3}, \frac{1}{6}\right) = 1 \cdot 2 - 4 < 0$

saddle

(j) Find the extreme values of $f(x,y) = x^2 + 2xy^2$ on the domain $x^2 + y^2 \leq 2$.

Critical points: $\nabla f = \langle 2x + 2y^2, 4xy \rangle = \bar{0}$

$$\Rightarrow \begin{cases} 2x+2y^2=0 \\ 4xy=0 \end{cases} \Rightarrow (0,0) \text{ is the only critical point.}$$

On the boundary: $g(x,y) = x^2+y^2 = 2 \Rightarrow$

$\nabla g = \langle 2x, 2y \rangle \Rightarrow \nabla f = \lambda \nabla g$

$$\langle 2x+2y^2, 4xy \rangle = \lambda \langle 2x, 2y \rangle$$

$$\Downarrow$$

$$\begin{cases} x+y^2 = \lambda x \\ 2xy = \lambda y \\ x^2+y^2 = 2 \end{cases} \Leftarrow \begin{cases} 2x+2y^2 = \lambda \cdot 2x \\ 4xy = \lambda \cdot 2y \\ x^2+y^2 = 2 \end{cases}$$

$$(2x-\lambda)y = 0 \Rightarrow \lambda = 2x \text{ or } y=0$$

$\lambda = 2x$:

$$\Downarrow$$

$$\begin{cases} x+y^2 = 2x^2 \\ x^2+y^2 = 2 \end{cases}$$

$$\Downarrow$$

$$x - x^2 = 2x^2 - 2$$

$$3x^2 - x - 2 = 0$$

$$x = \frac{1 \pm (1+24)^{1/2}}{6}$$

$$x = 1, -\frac{2}{3}$$

$y=0$:

$$\Downarrow$$

$$x = \lambda x$$

$$x^2 = 2$$

$$\Downarrow$$

$$x = \pm\sqrt{2}$$

If $x = 1 \Rightarrow 1^2 + y^2 = 2$

$$\Downarrow$$

$$y = \pm 1$$

If $x = -\frac{2}{3} \Rightarrow \frac{4}{9} + y^2 = 2$

$$y = \pm \frac{\sqrt{14}}{3}$$

$\Rightarrow$ points to check

	$(0,0)$;	$(\sqrt{2},0)$;	$(-\sqrt{2},0)$;	$(1,-1)$;	$(1,1)$;	$(-\frac{2}{3},-\frac{\sqrt{14}}{3})$;	$(-\frac{2}{3},\frac{\sqrt{14}}{3})$.
$f=$	0	2	2	3	3	$-\frac{44}{27}$	$-\frac{44}{27}$

max (for $(1,-1)$ and $(1,1)$)

min (for $(-\frac{2}{3},-\frac{\sqrt{14}}{3})$ and $(-\frac{2}{3},\frac{\sqrt{14}}{3})$)

(k) Find the maximum and minimum values of $f(x,y) = ye^x$ along the constraint curve $x^2 + 2y^2 = 4$.

$g(x,y) = x^2+2y^2 \qquad f(x,y) = ye^x$

$\nabla f = \langle e^x y, e^x\rangle \quad , \quad \nabla g = \langle 2x, 4y\rangle$

$$\Rightarrow \quad \nabla f = \lambda \nabla g \Rightarrow \begin{cases} e^x y = \lambda \cdot 2x \\ e^x = \lambda \cdot 4y \Rightarrow e^x > 0 \Rightarrow \lambda, y \neq 0 \\ x^2+2y^2 = 4 \end{cases}$$

Divide the first eq. by the second:

$$y = \frac{x}{2y} \Rightarrow 2y^2 = x \xRightarrow[\text{the last eqn}]{\text{subs into}} x^2+x=4$$

$$\Downarrow$$

$$x^2+x-4=0$$

$$\Downarrow$$

$$x = \frac{-1\pm(1+16)^{1/2}}{2} = \frac{-1\pm\sqrt{17}}{2}$$

because $x=2y^2$, only $x = \frac{-1+\sqrt{17}}{2}$ is a valid solution

$$\Rightarrow \quad y = \pm\sqrt{x/2} = \pm\sqrt{\frac{\sqrt{17}-1}{2}}$$

$\Rightarrow$ negative y is a minimum
positive y is a maximum

(1) Find the maximum value of $f(x, y) = x^2 - 6y^2 + 2$ along the constraint curve $2x + 3y = 5$.

$\nabla f = \langle 2x, -12y \rangle$; $\nabla g = \langle 2, 3 \rangle$

$\Rightarrow \langle 2x, -12y \rangle = \lambda \langle 2, 3 \rangle$

$$\begin{cases} 2x = 2\lambda \\ -12y = 3\lambda \\ 2x + 3y = 5 \end{cases} \Rightarrow \begin{cases} \left.\begin{array}{l} x = \lambda \\ -4y = \lambda \end{array}\right\} x = -4y \\ 2x + 3y = 5 \Rightarrow -8y + 3y = 5 \end{cases}$$

$\Downarrow$

$y = -1$

$\Downarrow$

$x = 4$

$f_{max} = f(4, -1) \Leftarrow$

$= 16 - 6 \cdot (-1)^2 + 2 = 12$