Part I: (1) a. false - partial derivatives need to be continuous for f to be differentiable - this info is not given.

b. false - f' is meaningless for a function of several variables.

c. True as can be checked by substituting.

(2) $2x^2+3y^2+z^2=24$ - level surface for $f(x,y,z)=2x^2+3y^2+z^2$ $\Rightarrow$ vector perpendicular to this surface is

$\nabla f = \langle 4x, 6y, 2z\rangle$ - this is also $\perp$ to the plane $8x-12y+4z=8$ if

$\langle 4x, 6y, 2z\rangle$ is $\parallel$ $\langle 8,-12,4\rangle$ or

$$\langle 4x, 6y, 2z\rangle = c\langle 8,-12,4\rangle$$

for some c to be determined. Then

$$4x=8c,\ 6y=-12c,\ 2z=4c \Rightarrow$$

$x = 2c,\ y = -2c,\ z = 2c$

This point must be on the surface

$2x^2 + 3y^2 + z^2 = 24 \Rightarrow 2\cdot(2c)^2 + 3(-2c)^2 + (2c)^2 = 24$

$\Rightarrow 24c^2 = 24 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1 \Rightarrow$

there are two points $(x,y,z) = (2,-2,2)$ and

$(x,y,z) = (-2,2,-2)$

Eq. of the tangent plane at $(2,-2,2)$

- vector perpendicular to this plane is still

$(8,-12,4)$ - is $8(x-2) - 12(y+2) + 4(z-2) = 0$

or $8x - 12y + 4z = 48$

(3) $f(x,y) = \frac{x}{y^2}$; $\frac{\partial f}{\partial x} = \frac{1}{y^2}$, $\frac{\partial f}{\partial y} = -\frac{2x}{y^3}$

- these functions are continuous near $(4,-2)$

$\Rightarrow f$ is differentiable at $(4,-2)$

$$f(x,y) \approx \underbrace{f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}_{\text{linearization at } (x_0,y_0) = \text{tangent plane at } (x_0,y_0)}$$

$$= \frac{4}{(-2)^2} + \frac{1}{(-2)^2}(x-4) - \frac{2\cdot 4}{(-2)^3}(y+2)$$

$$= \cancel{1} + \frac{1}{4}x - \cancel{1} + y - 2 = \frac{1}{4}x + y + 2$$

$$f(4.04, -1.99) = \frac{1}{4}(4.04) - 1.99 + 2 = 1.01 - 0.1 = 1.02$$

Part II.

(1) Find $D_{\bar{u}} f$ at P:

(a) $f(x,y) = (1+xy)^{3/2}$, $P(1,3)$, $\bar{u} = \frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}}\bar{j}$

unit vector

$$\nabla f(x,y) = \frac{3}{2}(1+xy)^{1/2}\langle y, x\rangle$$

$$\Rightarrow \nabla f(1,3) = \frac{3}{2}(1+1\cdot 3)^{1/2}\langle 3, 1\rangle = \langle 9, 3\rangle$$

$$D_{\bar{u}} f(1,3) = \langle 9,3\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \frac{12}{\sqrt{2}}$$

(b) $f(x,y,z) = \ln(x^2+2y^2+3z^2)$; $P(-1,2,4)$;

$\bar{u} = \left\langle -\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13} \right\rangle \to$ unit vector?

$\nabla f(x,y,z) = \frac{1}{x^2+2y^2+3z^2} \langle 2x, 4y, 6z \rangle$

$\nabla f(-1,2,4) = \frac{1}{1+8+48} \langle -2, 8, 24 \rangle$

$= \left\langle -\frac{2}{57}, \frac{8}{57}, \frac{24}{57} \right\rangle$

Check: $|\bar{u}|$

$= \left(\frac{9}{169} + \frac{16}{169} + \frac{144}{169} \right)^{1/2}$

$= 1$ ✓

$D_{\bar{u}} f(-1,2,4) = \left\langle -\frac{2}{57}, \frac{8}{57}, \frac{24}{57} \right\rangle \cdot \left\langle -\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13} \right\rangle$

$= \frac{6 - 32 - 288}{57 \cdot 13} = -\frac{314}{741}$

(2) • Find the directional derivative of $f(x, y) = \frac{x}{x+y}$ at $P(1, 0)$ in the direction of $Q(-1, 1)$.

$\nabla f = \left\langle \frac{x+y-x}{(x+y)^2}, -\frac{x}{(x+y)^2} \right\rangle = \left\langle \frac{y}{(x+y)^2}, -\frac{x}{(x+y)^2} \right\rangle$

$\nabla f(1,0) = \langle 0, -1 \rangle$; unit vector in the direction of Q: $n = \frac{\langle -2, 1 \rangle}{\sqrt{4+1}} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$

$\Rightarrow D_n f = \nabla f(1,0) \cdot n = \langle 0, -1 \rangle \cdot \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = -\frac{1}{\sqrt{5}}$

(3) Find the unit vector in the direction in which $f(x,y,z) = \frac{x+z}{z-y}$ decreases most rapidly at $P(5,7,6)$ and find the rate of

change of f in this direction.
The direction is opposite to that of

$\nabla f(5,7,6)$:

$$\nabla f(x,y,z) = \left\langle \frac{1}{z-y}, -\frac{x+z}{(z-y)^2}, \frac{z-y-(x+z)}{(z-y)^2} \right\rangle$$

$$= \left\langle \frac{1}{z-y}, -\frac{x+z}{(z-y)^2}, -\frac{x+y}{(z-y)^2} \right\rangle$$

$$\Rightarrow \nabla f(5,7,6) = \langle -1, -11, -12 \rangle$$

$\Rightarrow$ The direction of max decrease is

$$-\nabla f(5,7,6) = \langle 1, 11, 12 \rangle$$

max rate of decrease:

$$-|\nabla f(5,7,6)| = -(1^2+11^2+12^2)^{1/2} = \sqrt{266}.$$

(4)

- Let $r(x,y) = \sqrt{x^2+y^2}$.
  - Show that $\nabla r = \frac{\mathbf{r}}{r}$, where $\mathbf{r} = \langle x, y \rangle$.
  - Verify that $\mathbf{r}_1(t) = \langle x(t), y(t) \rangle = \langle \cos t, \sin t \rangle$ is a level set for $r(x,y)$ corresponding to $C = 1$.
  - Show that the gradient of r at $\mathbf{r}_1(t)$ is orthogonal to the level curve $\mathbf{r}_1(t)$ for any t.

- $r = (x^2+y^2)^{1/2} \Rightarrow \nabla r = \left\langle \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x, \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}y \right\rangle$

$$= \frac{\langle x, y \rangle}{(x^2+y^2)^{1/2}} = \frac{\bar{r}}{|\bar{r}|}$$

• $r(x(t), y(t)) = \sqrt{x^2(t)+y^2(t)} = \sqrt{\cos^2 t+\sin^2 t} = 1$

$\Rightarrow$ $(x(t), y(t))$ is the level curve for $c=1$.

• $\nabla r(x(t), y(t)) = \dfrac{\langle \cos t, \sin t\rangle}{\sqrt{\cos^2 t+\sin^2 t}} = \langle \cos t, \sin t\rangle$

Direction of the curve $F(t)$ is given by its tangent:

$F'(t) = \langle x'(t), y'(t)\rangle = \langle -\sin t, \cos t\rangle$

$\Rightarrow$ $\nabla r(x(t), y(t)) \cdot F'(t) = \cos t(-\sin t) + \sin t \cos t = 0.$

(5) Find all relative extrema and saddle points:

(a) $f(x,y) = xy - x^3 - y^3$

$\nabla f = \langle y-3x^2, x-3y^2\rangle = \langle 0,0\rangle$

$$\begin{cases} y-3x^2=0 \rightarrow y = 3x^2 \\ x-3y^2=0 \rightarrow x-3(3x^2)^2=0 \end{cases}$$

$x - 27x^4 = 0$

$x(1-27x^3)=0$

$x=0 \xrightarrow{y=3x^2} y=0$

$x = \dfrac{1}{\sqrt[3]{27}} = \dfrac{1}{3} \xrightarrow{y=3x^2} y = \dfrac{1}{3}$

Therefore, the critical points are

$$(0,0) \text{ and } \left(\frac{1}{3},\frac{1}{3}\right)$$

Classification: $f_{xx} = -6x$; $f_{xy} = 1$, $f_{yy} = -6y$

$\Rightarrow D(x,y) = f_{xx}f_{xy} - f_{xy}^2 = 36xy - 1$

At $(0,0)$: $D(0,0) = -1$ – saddle point

At $\left(\frac{1}{3},\frac{1}{3}\right)$: $f_{xx} = -6\cdot\frac{1}{3} = -2 < 0$

$D\left(\frac{1}{3},\frac{1}{3}\right) = 36\cdot\frac{1}{3}\cdot\frac{1}{3} - 1 = 3 > 0$

– this is a maximum.

(b) $g(x,z) = z\sin x$

$\nabla g(x,z) = \langle z\cos x, \sin x\rangle = \langle 0,0\rangle$

$\begin{cases} z\cos x = 0 \\ \sin x = 0 \end{cases} \rightarrow z = 0 \downarrow \sin x = 0 \downarrow x = \pi n$ or $\cos x = 0$ this cannot be combined with $\sin x = 0$

$\Rightarrow$ the critical points are $(\pi n, 0)$

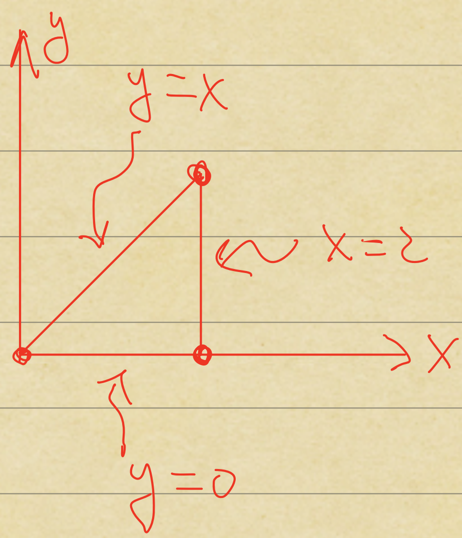
Classification: $f_{xx} = -z\sin x$, $f_{xz} = \cos x$, $f_{zz} = 0$

$\Rightarrow D(x,y) = f_{xx}f_{zz} - f_{xz}^2 = -\cos^2 x$

$\Rightarrow D(\pi n, 0) = -\cos^2(\pi n) = -1$ – saddle points

(6) Find the absolute extrema of

$f(x,y) = x^2 + 2y^2 - x$ on the triangular region with $(0,0)$, $(2,0)$, $(2,2)$:



(*) $\nabla f = \langle 2x-1, 4y \rangle = \langle 0,0 \rangle$

$\begin{cases} 2x-1=0 \\ y=0 \end{cases} \to \left(\frac{1}{2}, 0\right)$ is the critical point

$\Rightarrow \boxed{f\left(\frac{1}{2}, 0\right) = -\frac{1}{4}}$

(*) $y=0$, $0 \leq x \leq 2 \Rightarrow f(x,0) = x^2 - x$

↑ find extrema of this function on $[0,2]$

$\frac{d}{dx} f(x,0) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2} \Rightarrow$ need to test

$\boxed{f(0,0) = 0}$; $\quad f\left(\frac{1}{2}, 0\right)$; $\quad \boxed{f(2,0) = 2}$

– already computed

(*) $x=2,\ 0\le y\le 2 \Rightarrow f(2,y)=2+2y^2$

↑ find extrema of this function on $[0,2]$

$\frac{d}{dy} f(2,y) = 4y = 0 \Rightarrow y=0$

$\Rightarrow$ need to test:

$f(2,0)$; - already computed

$\boxed{f(2,2)=10}$

(*) $y=x,\ 0\le x\le 2 \Rightarrow f(x,x)=3x^2-x$

↑ find extrema of this function on $[0,2]$

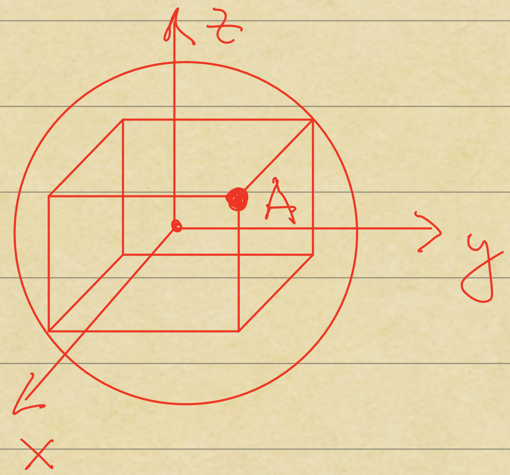
Already tested the endpoints, only need to find the critical point:

$\frac{d}{dx}(3x^2-x) = 6x-1=0 \Rightarrow x=\frac{1}{6} \Rightarrow$

$$\boxed{f\left(\frac{1}{6},\frac{1}{6}\right) = \frac{3}{36}-\frac{1}{6} = -\frac{1}{12}}$$

Hence $f\left(\frac{1}{2},0\right) = -\frac{1}{4}$ is the absolute minimum and $f(2,2)=10$ is the absolute maximum.

(7) Find the dimensions of the rectangular box of max. volume that can be inscribed in a sphere of radius a.



Dimensions of the box: $2x, 2y, 2z$

Volume of the box: $V(x,y,z) = 8xyz$

Vertex A with coordinates (x,y,z) lies on the sphere of radius $a \Rightarrow g(x,y,z) = x^2+y^2+z^2 = a^2$

$\nabla V = \langle 8yz, 8xz, 8xy \rangle$; $\nabla g = \langle 2x, 2y, 2z \rangle$

$$\nabla V = \lambda \nabla g \Rightarrow \begin{cases} 8yz = 2\lambda x \rightarrow 8xyz = 2\lambda x^2 \\ 8xz = 2\lambda y \rightarrow 8xyz = 2\lambda y^2 \\ 8xy = 2\lambda z \rightarrow 8xyz = 2\lambda z^2 \\ x^2+y^2+z^2 = a^2 \end{cases}$$

as $x, y, z, \lambda \neq 0$, because otherwise $V = 0$.

$\Rightarrow \quad 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2 \Rightarrow x^2 = y^2 = z^2$

$\Rightarrow x = y = z$, because $x > 0, y > 0, z > 0 \Rightarrow$

$x^2 + y^2 + z^2 = x^2 + x^2 + x^2 = a^2 \Rightarrow x = \frac{a}{\sqrt{3}}$

$\Rightarrow \quad x = y = z = \frac{a}{\sqrt{3}}$ and the maximum

volume is $V_{max} = \left(\frac{a}{\sqrt{3}}\right)^3$.