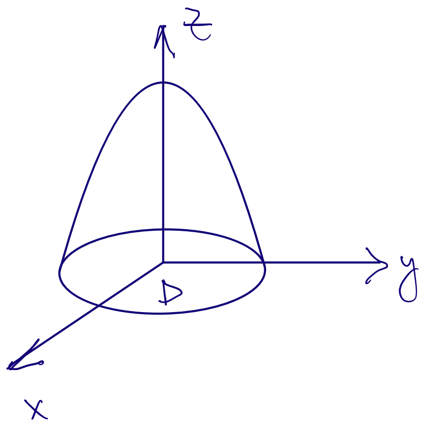
(d) Find the surface area of the portion of the surface $z = 10 - 2x^2 - 2y^2$ that lies above the xy-plane.



To find D: $z = 0 \Rightarrow$

$10 - 2x^2 - 2y^2 = 0 \Rightarrow$

$x^2 + y^2 = 5$ – D is a disk

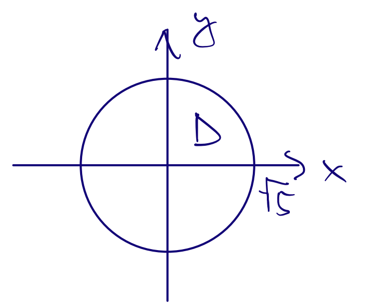
$$\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial y} = -2y$$

$$A = \iint_D \sqrt{z_x^2 + z_y^2 + 1}\, dA$$

$$= \iint_D (4x^2 + 4y^2 + 1)^{1/2}\, dA$$

use polar $= \int_0^{2\pi} \int_0^{\sqrt{5}} (4\rho^2\cos^2\theta + 4\rho^2\sin^2\theta + 1)^{1/2} \rho\, d\rho\, d\theta$



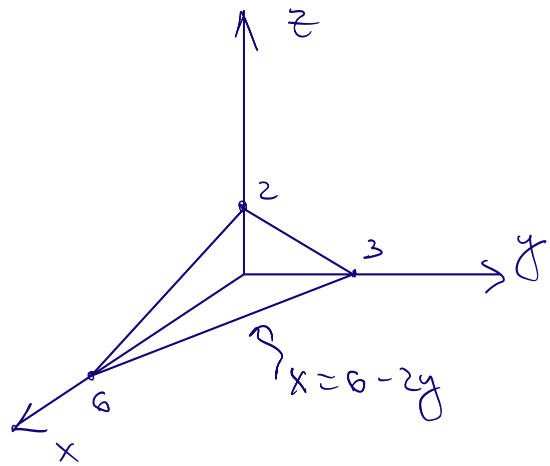
$$= 2\pi \int_0^{\sqrt{5}} (4\rho^2 + 1)^{1/2} \rho\, d\rho \underset{du = 8\rho\, d\rho}{\overset{u = 4\rho^2 + 1}{=}}$$

$D = [0, \sqrt{5}] \times [0, 2\pi]$ in polar coords

$$= 2\pi \int_1^{21} u^{1/2} \frac{1}{8}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^{21}$$

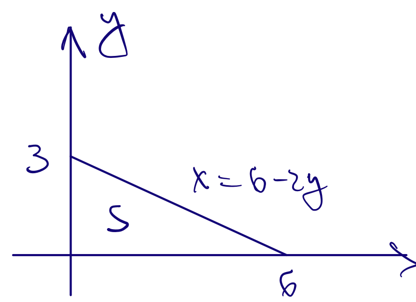
$$= \frac{\pi}{6}\left(21^{3/2} - 1\right)$$

(e) Evaluate $\iiint_D xy\,dV$, where D is the region in the first octant bounded by the coordinate planes and $x + 2y + 3z = 6$. $\longrightarrow$ $z = 2 - \frac{x}{3} - \frac{2y}{3}$



$z = 0 \Rightarrow x + 2y = 6 \Rightarrow$

$x = 6 - 2y$



$$\iiint_D xy\,dV = \iint_A \int_0^{2-x/3-2y/3} xy\,dz\,dA$$

$$= \int_0^3 \int_0^{6-2y} \int_0^{2-x/3-2y/3} xy\,dz\,dx\,dy = \int_0^3 \int_0^{6-2y} \left[xy \left(z \Big|_0^{2-\frac{x}{3}-\frac{2y}{3}} \right) \right] dx\,dy$$

$$= \int_0^3 \int_0^{6-2y} xy \left(2 - \frac{x}{3} - \frac{2y}{3} \right) dx\,dy = \int_0^3 \int_0^{6-2y} \left(2xy - \frac{1}{3}x^2 y - \frac{2}{3}xy^2 \right) dx\,dy = \int_0^3 \left[x^2 y - \frac{1}{9}x^3 y - \frac{1}{3}x^2 y^2 \right] \Big|_0^{6-2y} dy$$

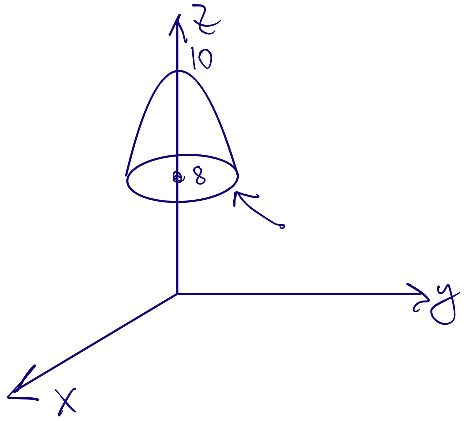
$$= \frac{1}{9} \int_0^3 \left[9(6-2y)^2 y - (6-2y)^3 y - 3(6-2y)^2 y^2 \right] dy$$

$$= \frac{1}{9} \int_0^3 \left[\cancel{324y} - \cancel{216y^2} + \cancel{36y^3} - \cancel{216y} + \cancel{216y^2} - \cancel{72y^3} + \cancel{8y^4} - \cancel{108y^2} + \cancel{36y^3} - \cancel{12y^4} \right] dy$$

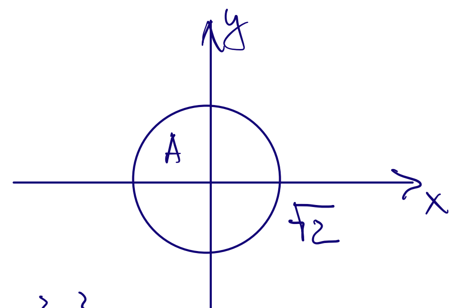
$$= \frac{1}{9}\int_0^3 \left[-4y^4 - 108y^2 + 108y\right] dy = \frac{1}{9}\left[-\frac{4}{5}y^5 - 36y^3 + 54y^2\right]\Big|_0^3$$

$$= \frac{1}{9}\left[-\frac{4}{5}\cdot 243 - 36\cdot 27 + 54\cdot 9\right]$$

(f) Evaluate $\displaystyle\iiint_D (x^2+y^2)^2\, dA$, where D is the region bounded above by $z = 10 - x^2 - y^2$ and below by $z = 8$.



$$10 - x^2 - y^2 = 8 \Rightarrow x^2 + y^2 = 2$$



$$\iiint_D (x^2+y^2)^2\, dV = \iint_A \int_0^{10-x^2-y^2} (x^2+y^2)^2\, dA \overset{\text{polar}}{=}$$

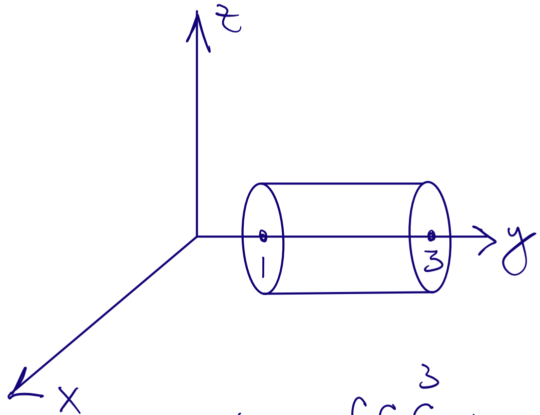
$$= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{10-\rho^2} \rho^4 \rho\, dz\, d\rho\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} \rho^5 \left(z\Big|_0^{10-\rho^2}\right) d\rho\, d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} \rho^5 (10-\rho^2)\, d\rho = 2\pi \int_0^{\sqrt{2}} 10\rho^5 - \rho^7\, d\rho$$

$$= 2\pi\left(\frac{5}{3}\rho^6 - \frac{1}{8}\rho^8\right)\Big|_0^{\sqrt{2}} = 2\pi\left(\frac{5}{3}(\sqrt{2})^6 - \frac{1}{8}(\sqrt{2})^8\right)$$

$$= 2\pi\left(\frac{5}{3}\cdot 8 - \frac{1}{8}\cdot 16\right) = \frac{68\pi}{3}$$

(g) Modify the standard cylindrical coordinates to evaluate $\iiint\limits_D (x^2+z^2)\,dA$, where D is the cylinder of radius 1 with axis centered along the y-axis, with $1 \le y \le 3$.



$x = \rho\cos\theta$

$y = y$

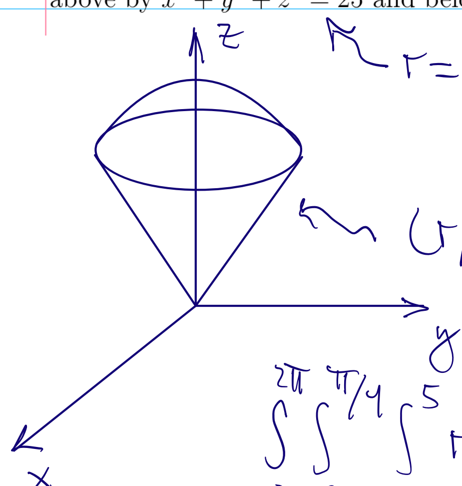
$z = \rho\sin\theta$

$$\Rightarrow \iint_A \int_1^3 (x^2+z^2)\,dy\,dA = \int_0^{2\pi}\int_0^1\int_1^3 (\rho^2\cos^2\theta + \rho^2\sin^2\theta)\,\rho\,dy\,d\rho\,d\theta = \int_0^{2\pi}\int_0^1\int_1^3 \rho^3\,dy\,d\rho\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^3\,d\rho \int_1^3 dy = 2\pi\cdot 2\cdot \frac{\rho^4}{4}\Big|_0^1 = \pi$$

(h) Use spherical coordinates to evaluate $\iiint\limits_D \underbrace{x^2+y^2+z^2}_{r^2}\,dA$, where D is the region bounded above by $x^2+y^2+z^2 = 25$ and below by $z = \sqrt{x^2+y^2}$.

$r = 5$ $\qquad \varphi = \frac{\pi}{4}$



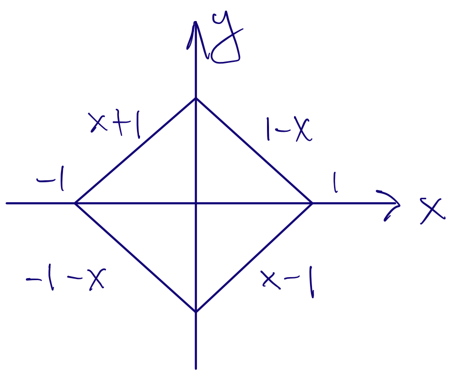
$(r,\theta,\varphi) \in [0,5]\times[0,2\pi]\times[0,\frac{\pi}{4}]$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^5 r^2\cdot r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$= 2\pi \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^5 r^4 \, dr = 2\pi \left(-\cos\varphi \Big|_0^{\pi/4} \right) \left(\frac{r^5}{5} \Big|_0^5 \right)$$

$$= 2\pi \left(1 - \frac{\sqrt{2}}{2} \right) \cdot 625$$

(i) Use a change of variables to evaluate $\displaystyle\iint_D xy\, dA$, where D is the diamond-shaped domain bounded by $y = x + 1$, $y = 1 - x$, $y = x - 1$ and $y = -1 - x$.



the lines are $y - x = 1 \quad y - x = -1$

$\qquad\qquad\qquad y + x = 1 \quad y + x = -1$

set $u = y - x$, $v = y + x \Rightarrow u \in [-1, 1]$, $v \in [-1, 1]$

Jacobian: $\left| \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \right| = \left| \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \right| = 2$

also, $u + v = 2y$, $v - u = 2x \Rightarrow x = \frac{v-u}{2}$, $y = \frac{u+v}{2}$

$\Rightarrow xy = \frac{v^2 - u^2}{4}$

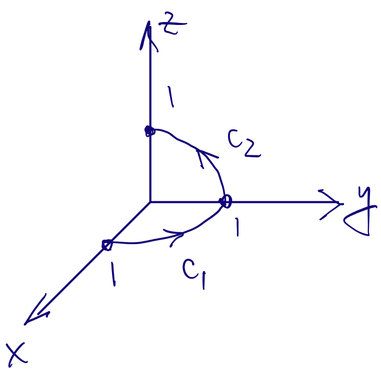
$$\Rightarrow \iint_D xy\, dA = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} (v^2 - u^2)\, 2 \, du\, dv$$

$$= \frac{1}{2}\int_{-1}^{1}\left(v^2u - \frac{u^3}{3}\right)\Big|_{-1}^{1} dv = \frac{1}{2}\int_{-1}^{1}\left(v^2 - \frac{1}{3} + v^2 - \frac{1}{3}\right) dv$$

$$= \frac{1}{2}\int_{-1}^{1}\left(2v^2 - \frac{2}{3}\right) dv = \left(\frac{v^3}{3} - \frac{1}{3}v\right)\Big|_{-1}^{1}$$

$$= \left(\frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3}\right) = 0$$

(a) Evaluate $I = \int_C x + 2y + 3z\, ds$, where $C = C_1 + C_2$. The first segment of the path, C_1, is the arc of the circle $x^2 + y^2 = 1$ lying in the xy-plane, beginning at the point $(1,0,0)$ and ending at the point $(0,1,0)$. The second segment of the path, C_2, is the arc of the circle $y^2 + z^2 = 1$ lying in the yz-plane, beginning at the point $(0,1,0)$ and ending at the point $(0,0,1)$.



c_1: $\bar{r}(\theta) = \langle \cos\theta, \sin\theta, 0\rangle$

$\theta \in [0, \frac{\pi}{2}]$

c_2: $\bar{r}(\theta) = \langle 0, \cos\theta, \sin\theta\rangle$

$\theta \in [0, \frac{\pi}{2}]$

$$\int_{C_1} (x+2y+3z)\, ds = \int_0^{\pi/2} (\cos\theta + 2\sin\theta)\left((-\sin\theta)^2 + \cos^2\theta\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} (\cos\theta + 2\sin\theta)\, d\theta = (\sin\theta - 2\cos\theta)\Big|_0^{\frac{\pi}{2}} = (1-(-2)) = 3$$

$$\int_{C_1} (x+2y+3z)\, ds = \int_0^{\pi/2} (2\cos\theta + 3\sin\theta)\left((-\sin\theta)^2 + \cos^2\theta\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} (2\cos\theta + 3\sin\theta)\, d\theta = (2\sin\theta - 3\cos\theta)\Big|_0^{\frac{\pi}{2}} = (2-(-3)) = 5$$

$$\Rightarrow \int_C (x+2y+3z)\, ds = 5 + 3 = 8$$

(b) Find the work done (the vector line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$) by a particle moving through the vector field $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} - x\mathbf{k}$ along the path $C = C_1 + C_2$, where C_1 is the straight line segment from $(1, 1, 1)$ to $(2, 3, 4)$ and C_2 is the straight line segment from $(2, 3, 4)$ to $(0, -1, 5)$.

C_1: $\vec{r}(t) = \langle 1,1,1\rangle + t(\langle 2,3,4\rangle - \langle 1,1,1\rangle)$

$= \langle 1,1,1\rangle + t\langle 1,2,3\rangle = \langle 1+t, 1+2t, 1+3t\rangle$

$t \in [0,1]$

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 \langle 1+2t, (1+3t)^2, -(1+t)\rangle$

$\cdot \langle 1,2,3\rangle\,dt = \int_0^1 (1+2t+2(1+3t)^2 - 3(1+t))\,dt$

$= \int_0^1 (1+\cancel{2t}+2+\cancel{12t}+\cancel{18t}^2-3-\cancel{3t})\,dt = \int_0^1 (18t^2+11t)\,dt$

$= \left(6t^3 + \frac{11t^2}{2}\right)\Big|_0^1 = \frac{23}{2}$

C_2: $\vec{r}(t) = \langle 2,3,4\rangle + t(\langle 0,-1,5\rangle - \langle 2,3,4\rangle)$

$= \langle 2-2t, 3-4t, 4+t\rangle$, $t \in [0,1]$

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 3-4t, (4+t)^2, 2t-2\rangle \cdot \langle -2,-4,1\rangle\,dt$

$= \int_0^1 (8t-6-4(4+t)^2+2t-2)\,dt$

$= \int_0^1 (10t-8-64-32t-4t^2)\,dt = \int_0^1 (-4t^2-22t-72)\,dt$

$= \left(-\frac{4}{3}t^3 - 11t^2 - 72t\right)\Big|_0^1 = -\frac{4}{3} - 11 - 72 = -84\frac{1}{3}$

$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \frac{23}{2} - 84\frac{1}{3}$

(c) Show that the vector field
$\mathbf{F} = \langle \underbrace{e^y + \sec x \tan x \tan y + y\cos(xy) + 2x}_{P}, \underbrace{xe^y + \sec x \sec^2 y + x\cos(xy) - 3y^2}_{Q} \rangle$ is conservative, and find a potential for it.

$$\frac{\partial Q}{\partial x} = e^y + \tan x \sec x \sec^2 y + \cos xy - xy\sin(xy)$$

$$\parallel$$

$$\frac{\partial P}{\partial y} = e^y + \sec x \tan x \sec^2 y + \cos xy - xy\sin(xy)$$

– conservative

$$\varphi = \int P\,dx = \int (e^y + \sec x \tan x \tan y + y\cos(xy) + 2x)\,dx$$

$$= xe^y + \sec x \tan y + \sin(xy) + x^2 + C(y)$$

$$xe^y + \sec x \sec^2 y + x\cos xy - 3y^2 = Q = \frac{\partial \varphi}{\partial y} = xe^y + \sec x \sec^2 y + x\cos(xy) + C'(y)$$

$$\Rightarrow C'(y) = -3y^2 \Rightarrow C(y) = -y^3$$

$$\Rightarrow \varphi = xe^y + \sec x \tan y + \sin(xy) + x^2 - y^3$$

(d) The vector field $\mathbf{F} = \left\langle y^2z^4 + \frac{1}{x} + \frac{z}{y}e^{xz}, 2xyz^4 + \frac{1}{y} - \frac{1}{y^2}e^{xz}, 4xy^2z^3 - \frac{1}{z} + \frac{x}{y}e^{xz} \right\rangle$ is conservative (you don't need to verify it). Find a potential for $\mathbf{F}$.

$$\varphi = \int \left(y^2z^4 + \frac{1}{x} + \frac{z}{y}e^{xz}\right)dx = xy^2z^4 + \ln|x| + \frac{1}{y}e^{xz} + C(y,z)$$

$$\cancel{2xyz^4} + \frac{1}{y} - \cancel{\frac{1}{y^2}e^{xz}} = Q = \frac{\partial \varphi}{\partial y} = \cancel{2xyz^4} - \cancel{\frac{1}{y^2}e^{xz}} + \frac{\partial C}{\partial y}$$

$$\Rightarrow \frac{\partial C}{\partial y} = \frac{1}{y} \Rightarrow C(y,z) = \int \frac{1}{y}\,dy = \ln|y| + D(z)$$

$$\Rightarrow \varphi = xy^2z^4 + \ln|x| + \frac{1}{y}e^{xz} + \ln|y| + D(z)$$

$$\cancel{4xy^2z^3} - \frac{1}{z} + \cancel{\frac{x}{y}e^{xz}} = R = \frac{\partial \varphi}{\partial z} = \cancel{4xy^2z^3} + \cancel{\frac{x}{y}e^{xz}} + D'(z)$$

$$\Rightarrow D'(z) = -\frac{1}{z} \Rightarrow D(z) = -\ln z$$

$$\varphi = xy^2z^4 + \ln|x| + \frac{1}{y}e^{xz} + \ln|y| - \ln z$$

(e) Use Green's Theorem to evaluate the vector line integral of $\mathbf{F} = \left\langle \frac{y^2}{2} + \tan(e^{x^2}), \ln(y^4) - \frac{x^2}{2} \right\rangle$ over the closed curve C, which is the circle of radius 2 centered at the origin oriented in the counterclockwise direction.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \iint_D (-x - y)\, dA$$

$$= -\int_0^{2\pi} \int_0^2 (\rho\cos\theta + \rho\sin\theta)\,\rho\, d\rho\, d\theta$$

$$= -\underbrace{\int_0^{2\pi} (\cos\theta + \sin\theta)\, d\theta}_{0} \int_0^2 \rho^2\, d\rho = 0$$

(f) Find the divergence and curl of $\mathbf{F} = \sin(x^2 + y)\mathbf{i} - e^{xz}\mathbf{j} + \ln(x + 2y + 3z)\mathbf{k}$.

$$\operatorname{div} F = \frac{\partial}{\partial x}\sin(x^2+y) + \frac{\partial}{\partial y}\left(-e^{xz}\right) + \frac{\partial}{\partial z}\ln(x+2y+3z)$$

$$= \cos(x^2+y)\cdot 2x + 0 + \frac{3}{x+2y+3z}$$

$$\text{curl}\,F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(x^2+y) & -e^{xz} & \ln(x+2y+3z) \end{vmatrix}$$

$$= \left\langle \frac{2}{x+2y+3z} + xe^{xz},\ -\frac{1}{x+2y+3x},\ -ze^{xz} - \cos(x^2+y) \right\rangle$$