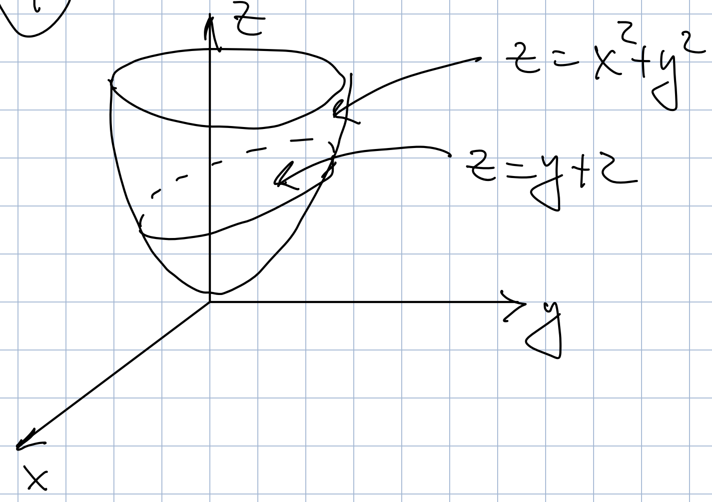
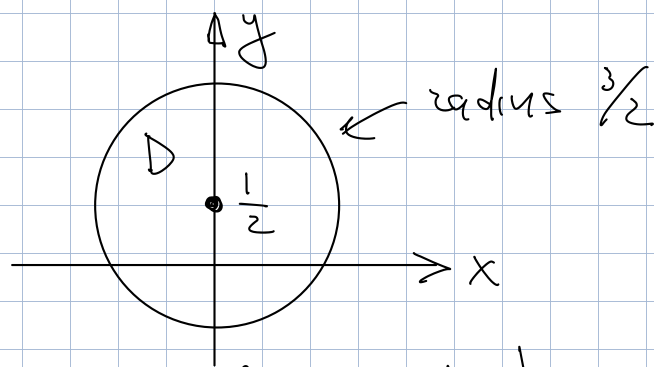
1



Intersection: $x^2+y^2=y+2 \Rightarrow$

$x^2+y^2-y+\frac{1}{4}=2+\frac{1}{4} \Rightarrow x^2+(y-\frac{1}{2})^2=\frac{9}{4}$



Use polar coordinates

$x=r\cos\theta,\ y=\frac{1}{2}+r\sin\theta$

$\Rightarrow$ equation of the circle is $r=\frac{3}{2}$; also,

- $y+2=r\sin\theta+\frac{5}{2}$
- $x^2+y^2=r^2\cos^2\theta+\frac{1}{4}+r\sin\theta+r^2\sin^2\theta=r^2+r\sin\theta+\frac{1}{4}$

$$\text{Volume}=\iint_D\int_{x^2+y^2}^{y+2} dz\,dA$$

$$=\int_0^{3/2}\int_0^{2\pi}\int_{r^2+r\sin\theta+\frac{1}{4}}^{r\sin\theta+5/2} r\,dz\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{3/2}\int_0^{2\pi}\left[\left(r\sin\theta+\frac{5}{2}\right)^2-\left(r^2+r\sin\theta+\frac{1}{4}\right)^2\right]dr$$

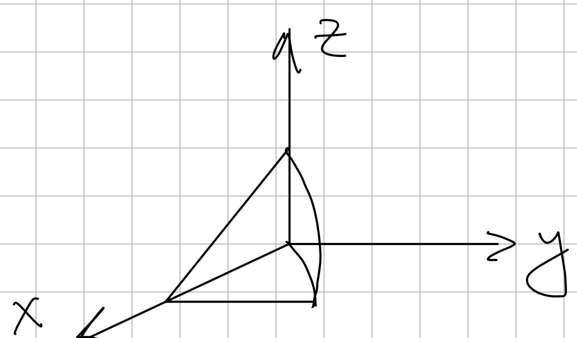
$$=\frac{1}{2}\int_0^{3/2}\int_0^{2\pi}\Big[\cancel{r^2\sin^2\theta}+\underline{5r\sin\theta}+\frac{25}{4}-r^4-\cancel{r^2\sin^2\theta}-\frac{1}{16}-\underline{2r^3\sin\theta}-\frac{1}{2}r^2$$
$$-\frac{1}{2}\underline{r\sin\theta}\Big]\,d\theta\,dr$$

Here underlined integrate to 0 in θ.

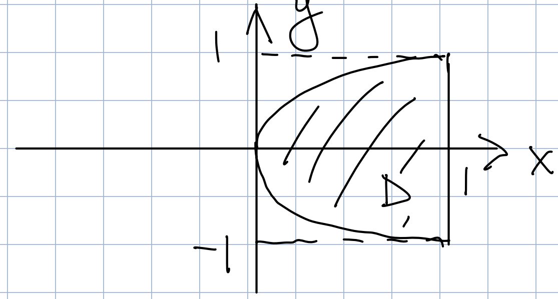
$$=\frac{1}{2}\int_0^{3/2}\int_0^{2\pi}\left(\frac{99}{16}-\frac{1}{2}r^2-r^4\right)d\theta\,dr=\pi\int_0^{3/2}\left(\frac{99}{16}-\frac{1}{2}r^2-r^4\right)dr$$

$$=\pi\left(\frac{99}{16}r-\frac{1}{6}r^3-\frac{r^5}{5}\right)\Big|_0^{3/2}=\pi\left(\frac{297}{32}-\frac{27}{48}-\frac{243}{160}\right)=\frac{36\pi}{5}$$

2



+ reflected piece in the fourth octant



$$\text{Volume} = \iiint_V dV = \iint_D \int_0^{1-x} dz\, dA$$

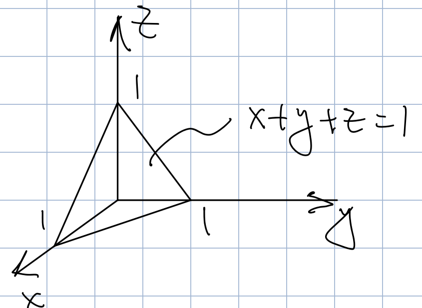
$$= \int_{-1}^{1}\int_{y^2}^{1}\int_0^{1-x} dz\,dx\,dy = \int_{-1}^{1}\int_{y^2}^{1} (1-x)\,dx\,dy$$

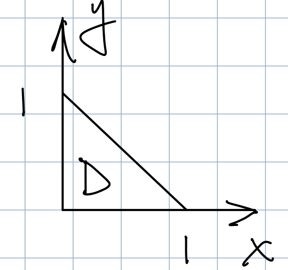
$$= \int_{-1}^{1} \left(x - \tfrac{1}{2}x^2\right)\Big|_{y^2}^{1} dy = \int_{-1}^{1}\left(\tfrac{1}{2} - y^2 + \tfrac{1}{2}y^4\right) dy$$

$$= \tfrac{1}{2}y - \frac{y^3}{3} + \tfrac{1}{10}y^5 \Big|_{-1}^{1} = 2\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{10}\right) = \frac{8}{15}$$

(3)

$$\iiint_T x^2\, dA = \iint_D \int_0^{1-x-y} x^2\, dz\, dA = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\, dz\,dy\,dx$$





$$= \int_0^1\int_0^{1-x} x^2(1-x-y)\,dy\,dx$$

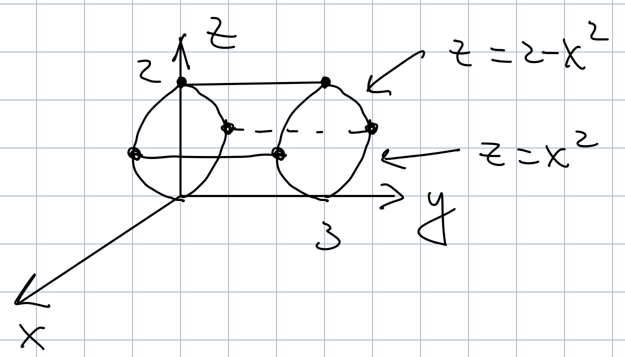
$$= \int_0^1\int_0^{1-x} (x^2 - x^3 - x^2 y)\,dy\,dx$$

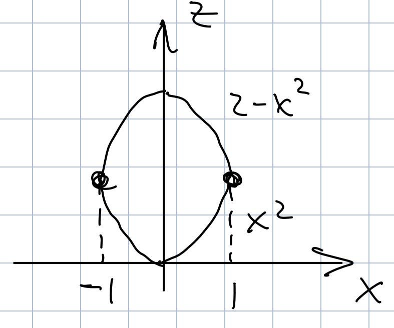
$$= \int_0^1 \left(x^2y - x^3y - \tfrac{1}{2}x^2y^2\right)\Big|_0^{1-x} dx = \int_0^1 \left(x^2(1-x) - x^3(1-x) - \tfrac{1}{2}x^2(1-x)^2\right) dx$$

$$= \int_0^1 \left(x^2 - x^3 \cancel{- x^3} + x^4 - \tfrac{1}{2}x^2 \cancel{+ x^3} - \tfrac{1}{2}x^4\right) dx = \int_0^1 \left(\tfrac{1}{2}x^4 - x^3 + \tfrac{1}{2}x^2\right) dx$$

$$= \left(\tfrac{1}{10}x^5 - \tfrac{1}{4}x^4 + \tfrac{1}{6}x^3\right)\Big|_0^1 = \tfrac{1}{10} - \tfrac{1}{4} + \tfrac{1}{6} = \tfrac{1}{60}$$

(4)





$$2 - x^2 = x^2$$

$$2x^2 = 2$$

$$\Rightarrow x = \pm 1$$

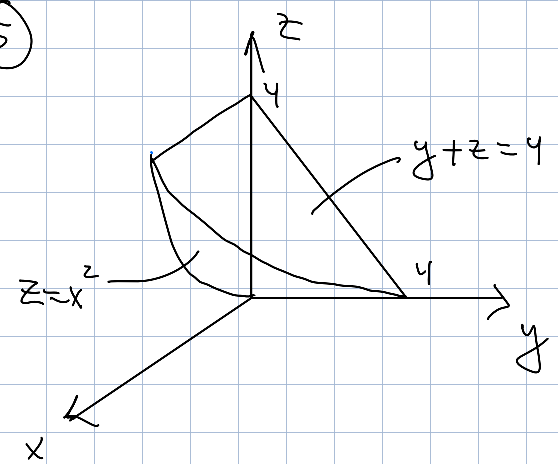
$$\iiint_T (x+y)\,dV = \iint_D \int_0^3 (x+y)\,dy\,dA = \int_{-1}^{1}\int_{x^2}^{2-x^2}\int_0^3 (x+y)\,dy\,dz\,dx$$

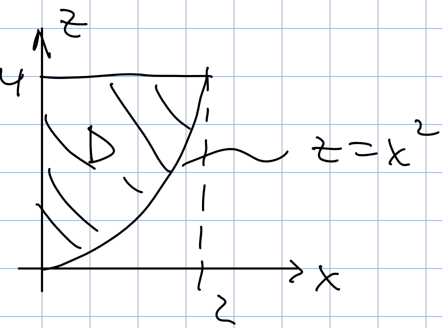
$$= \int_{-1}^{1}\int_{x^2}^{2-x^2} \left(xy+\frac{y^2}{2}\right)\Big|_0^3 dz\,dx = \int_{-1}^{1}\int_{x^2}^{2-x^2}\left(3x-\frac{9}{2}\right)dz\,dx$$

$$= \int_{-1}^{1}\left(3xz-\frac{9}{2}z\right)\Big|_{x^2}^{2-x^2} dx = \int_{-1}^{1}\Big[\underbrace{3x(2-2x^2)}_{\text{odd f-n integrates to 0 on }(-1,1)} - \frac{9}{2}(2-2x^2)\Big]dx$$

$$= -9\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = -9\left(1-\frac{1}{3}\right)+9\left(-1+\frac{1}{3}\right) = -12$$

5





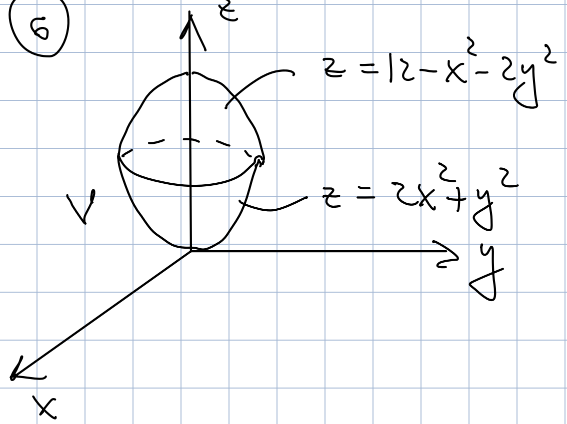
$$\iiint_V x\,dV = \iint_A \int_0^{4-z} x\,dy\,dA = \int_0^2\int_{x^2}^{4}\int_0^{4-z} x\,dy\,dz\,dx$$

$$= \int_0^2 x\int_{x^2}^{4}(4-z)\,dz\,dx = \int_0^2 x\left(4z-\frac{z^2}{2}\right)\Big|_{x^2}^{4} dx = \int_0^2 x\left(8-4x^2+\frac{1}{2}x^4\right)dx$$

$$= \int_0^2 \left(8x-4x^3+\frac{1}{2}x^5\right)dx = \left(4x^2-x^4+\frac{1}{12}x^6\right)\Big|_0^2$$

$$= 16-16+\frac{64}{12} = \frac{16}{3}.$$

(6)

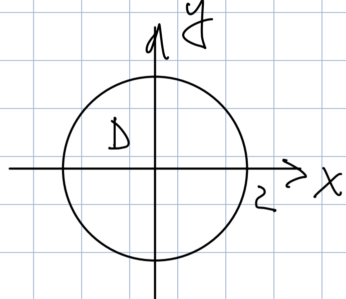


Intersection:

$$12 - x^2 - 2y^2 = 2x^2 + y^2$$

$$\Rightarrow 3x^2 + 3y^2 = 12 \Rightarrow x^2 + y^2 = 4$$

$\Rightarrow$



In cylindrical coordinates:

$$0 \leq r \leq 2$$
$$0 \leq \theta \leq 2\pi$$

and $2x^2 + y^2 = 2r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2(2\cos^2\theta + \sin^2\theta)$

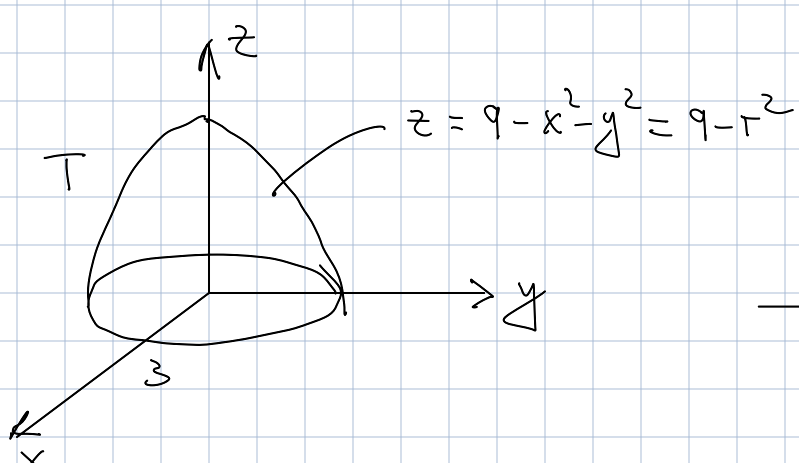
$12 - x^2 - 2y^2 = 12 - r^2\cos^2\theta - 2r^2\sin^2\theta$
$= 12 - r^2(\cos^2\theta + 2\sin^2\theta)$

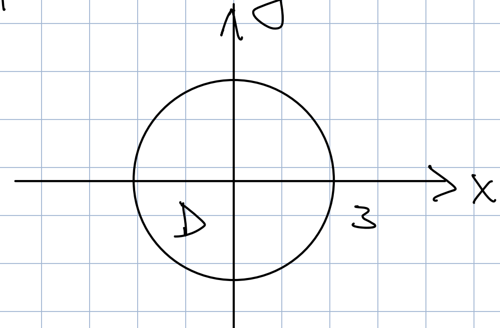
$$\text{Volume} = \iiint_V dV = \iint_D \int_{2x^2+y^2}^{12-x^2-2y^2} dz\, dA = \iint_D \int_{r^2(2\cos^2\theta+\sin^2\theta)}^{12-r^2(\cos^2\theta+2\sin^2\theta)} dz \quad dA$$

$$= \int_0^{2\pi}\int_0^2 \int_{r^2(2\cos^2\theta+\sin^2\theta)}^{12-r^2(\cos^2\theta+2\sin^2\theta)} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\Big[\underbrace{12-r^2(\cos^2\theta+2\sin^2\theta) - r^2(2\cos^2\theta+\sin^2\theta)}_{=12-3r^2}\Big]\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (12r - 3r^3)\, dr\, d\theta = 2\pi\left(6r^2 - \frac{3}{4}r^4\right)\Big|_0^2 = 24\pi$$

(7)





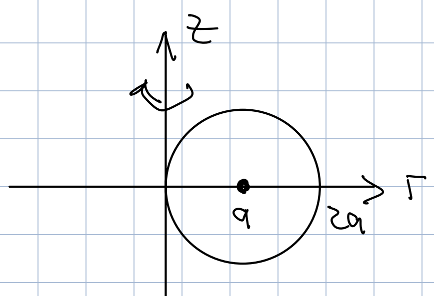
$$\iiint_T z\,dV = \iint_D \int_0^{9-r^2} z\,dz\,dA = \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} z\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r\left(\frac{z^2}{2}\Big|_0^{9-r^2}\right)dr\,d\theta = \pi\int_0^3 r\,(9-r^2)^2\,dr \underset{du=-2r\,dr}{\overset{u=9-r^2}{=}}$$

$$= \frac{\pi}{2}\int_0^9 u^2\,du = \frac{\pi}{2}\,\frac{u^3}{3}\Big|_0^9 = \frac{243\pi}{2}$$

(8) Recall that $\rho\sin\varphi = r \Rightarrow$ multiply $\rho = 2a\sin\varphi$ by $\rho \Rightarrow \rho^2 = 2a\rho\sin\varphi \Rightarrow \underbrace{x^2+y^2}_{r^2}+z^2 = 2ar \Rightarrow r^2+z^2 = 2ar$

$$\Rightarrow r^2 - 2ar + a^2 + z^2 = a^2 \Rightarrow (r-a)^2 + z^2 = a^2$$



$\Rightarrow$ solid is a donut w/o hole.

$$\text{Volume} = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2a\sin\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\sin\varphi\left(\rho^3\Big|_0^{2a\sin\varphi}\right)d\varphi\,d\theta = \frac{8a^3}{3}\int_0^{2\pi}\int_0^{\pi}\sin^4\varphi\,d\varphi\,d\theta$$

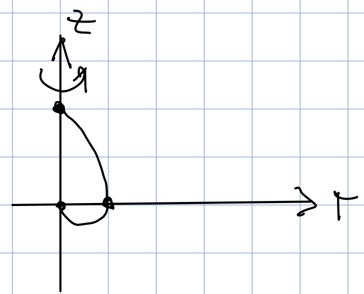
$$= \frac{16\pi a^3}{3}\int_0^{\pi}\sin^4\varphi\,d\varphi = \frac{2\pi a^3}{3}\int_0^{\pi}(3 - 4\cos 2\varphi + \cos 4\varphi)\,d\varphi$$

$$= \frac{2\pi a^3}{3}\left(3\varphi - 2\sin 2\varphi + \frac{1}{4}\sin 4\varphi\right)\Big|_0^{\pi} = 2\pi^2 a^3$$

since: $8\sin^4\varphi = 2(2\sin^2\varphi)^2 = 2(1-\cos 2\varphi)^2 = 2 - 4\cos 2\varphi + 2\cos^2 2\varphi$

$$= 2 - 4\cos 2\varphi + 1 + \cos 4\varphi = 3 - 4\cos 2\varphi + \cos 4\varphi$$

(9) $\rho = 1+\cos\varphi$: θ is missing $\Rightarrow$ the surface is symmetric w.r.t. z-axis. Plot this in rz-plane:



$$\rho(0) = 2 \;;\; \rho\left(\tfrac{\pi}{2}\right) = 1 \;;\; \rho(\pi) = 0$$

$$\text{Volume} = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \left(\rho^3 \Big|_0^{1+\cos\varphi}\right) d\varphi \, d\theta = \frac{2\pi}{3}\int_0^{\pi} (1+\cos\varphi)^3 \sin\varphi \, d\varphi$$

$$\underset{du = -\sin\varphi\, d\varphi}{\overset{u = 1+\cos\varphi}{=}} -\frac{2\pi}{3}\int_2^0 u^3 \, du = -\frac{2\pi}{3}\frac{u^4}{4}\Big|_2^0 = \frac{16\pi}{3}$$

(10)
$$\begin{cases} u = x - 2y \\ v = 3x + y \end{cases} \Rightarrow \begin{array}{l} v - 3u = (\cancel{3x} + y) - 3(\cancel{x} - 2y) = 7y \\ u + 2v = x - \cancel{2y} + 6x + \cancel{2y} = 7x \end{array}$$

$$\begin{cases} x = \frac{1}{7}u + \frac{2}{7}v \\ y = -\frac{3}{7}u + \frac{1}{7}v \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} 1/7 & -3/7 \\ \frac{2}{7} & \frac{1}{7} \end{vmatrix} \right| = \left| \frac{1}{14} + \frac{6}{14} \right| = \frac{1}{2}$$

(11)
$$\begin{array}{l} 1 \le x + y \le 2 \\ 2 \le 2x - 3y \le 3 \end{array} \Rightarrow \begin{array}{l} 1 \le u \le 2 \\ 2 \le v \le 3 \end{array}$$

$$\begin{cases} u = x + y \\ v = 2x - 3y \end{cases} \Rightarrow \begin{array}{l} 2u - v = \cancel{2x} + 2y - \cancel{2x} + 6y = 8y \\ 3u + v = 3x + \cancel{3y} + 2x - \cancel{3y} = 5x \end{array}$$

$$\Rightarrow \begin{cases} x = \frac{3}{5}u + \frac{1}{5}v \\ y = \frac{1}{4}u - \frac{1}{8}v \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} \frac{3}{5} & \frac{1}{4} \\ \frac{1}{5} & -\frac{1}{8} \end{vmatrix} \right| = \frac{1}{10}$$

$$\text{Area} = \iint_A dA = \int_1^2 \int_2^3 \frac{1}{10}\, du\, dv = \frac{1}{10}$$

(12) The region is between $y = x$ and $y = 2x$

$\Rightarrow$ it is between $\frac{y}{x} = 1$ and $\frac{y}{x} = 2$

$\Rightarrow$ $v = \frac{y}{x}$ satisfies $1 \le v \le 2$

The region is between $xy = 1$ and $xy = 2$

$\Rightarrow$ $u = xy$ satisfies $1 \le u \le 2$

$$uv = xy \cdot \frac{y}{x} = y^2 \Rightarrow \boxed{y = \sqrt{uv}}$$

$$u/v = xy \cdot \frac{x}{y} = x^2 \Rightarrow \boxed{x = \sqrt{u/v}}$$

Then:

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{u/v}} \cdot \frac{1}{v} = \frac{1}{2\sqrt{uv}}; \quad \frac{\partial x}{\partial v} = \frac{1}{2\sqrt{u/v}}\left(-\frac{u}{v^2}\right) = -\frac{u}{v}\frac{1}{2\sqrt{uv}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2\sqrt{uv}} \cdot v \; ; \quad \frac{\partial y}{\partial v} = \frac{1}{2\sqrt{uv}}\, u$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{u}{v}\frac{1}{2\sqrt{uv}} \\ \frac{1}{2\sqrt{uv}}\, v & \frac{1}{2\sqrt{uv}}\, u \end{vmatrix} \right| = \frac{\cancel{u}}{4\cancel{u}v} + \frac{\cancel{u}}{4\cancel{u}v}$$

$$= \frac{1}{2v}$$

$$\Rightarrow \text{Area} = \iint_A dA = \int_1^2 \int_1^2 \frac{1}{2v}\, dv\, du = \frac{1}{2} \ln v \Big|_1^2 = \frac{1}{2}\ln 2.$$